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Imitation in Heterogeneous Populations

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Abstract

We study a boundedly rational model of imitation when payoff distributions of actions differ across types of individuals. Individuals observe others' actions and payoffs, and a comparison signal. One of two inefficiencies always arises: (i) *uniform adoption*, i.e., all individuals choose the action that is optimal for one type but sub-optimal for the other, or (ii) *dual incomplete learning*, i.e., only a fraction of each type chooses its optimal action. Which one occurs depends on the composition of the population and how critical the choice is for different types of individuals. In an application, we show that a monopolist serving a population of boundedly rational consumers cannot fully extract the surplus of high-valuation consumers, but can sell to consumers who do not value the good.

Key words: Imitation, heterogeneous populations, bounded rationality, Fubini extension.

JEL codes: D81, D83.

1 Introduction

We study decision-making when individuals have little information about the outcome of choosing among different actions. Although individuals may use information from their own experience, they may also use

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information obtained by observing other individuals.[1] In many situations, however, it is unlikely that the same choice yields similar outcomes to different individuals. For instance, if a farmer observes another farmer from a different geographic area, who obtains high profits after adopting hybrid maize, this may suggest that she should adopt hybrid maize as well. Yet, she may intuit that she might not reach such a high profit if her farm is located in a remote area with poor infrastructure, where obtaining hybrid seeds and fertilizers is much more costly.[2] The literature on boundedly rational learning and imitation has paid little attention to how individuals learn in the presence of heterogeneity. In particular, do all individuals converge to the same action or does each individual converge to her optimal choice? What factors determine the actions that are chosen in the long run? This paper offers answers to these and other related questions.

Our framework is similar to the setup of boundedly rational learning in social contexts (e.g., Ellison and Fudenberg, 1993, 1995; Schlag 1998). Here individuals learn from their own and others' experiences in a process of repeated decision making. Hence, these models provide a natural environment for analyzing the implications of heterogeneity. We consider a population of individuals of two different, fixed and exogenously given, types (A and B) who repeatedly choose between two actions (a and b). Payoffs are determined according to distribution functions that are unknown to the individuals and that depend on the individual's type and chosen action. The expected payoff of a (b) is greater than the expected payoff of b (a) for type A (B) individuals. An individual makes choices by considering only three pieces of information: (i) her own most recent choice and consequent payoff, (ii) the most recent choice and payoff of a randomly sampled individual in the population, and (iii) a random *comparison signal*. The comparison signal an individual observes is informative about the difference between the individual's and the observed individual's expected payoff associated with the observed individual's choice. In our example of the farmers above, the comparison signal originates from the information they may have about the difference in costs of obtaining seeds and fertilizer for hybrid maize. This information suggests to a remotely located farmer that she may not obtain as high profits as a farmer located in a geographic area with better infrastructure, if she follows him in adopting hybrid maize. Whenever an individual observes an action different from the one that she is currently using, she makes a new choice using a *decision rule*. The decision rule is a mapping from the payoffs obtained by herself and the observed individual, and the observed comparison signal, to the probability of switching to the action of the observed individual. We confine our analysis to a class of decision rules that are linear in observed payoffs and the comparison signal. We show (Proposition 1) that linearity and the unbiasedness of comparison signals combined yield *payoff-ordering*, i.e., on average, given that an individual observes the two actions, she is most likely to make the right choice.[3]

[1]The role of imitation has been widely documented both empirically and experimentally (e.g., Munshi 2004 and Apesteguia, Huck, Oechssler 2007, respectively). See also Young (2009).

[2]This example is inspired in the analysis of hybrid maize adoption among heterogeneous farmers in Kenya (Suri 2011).

[3]See, e.g., Schlag (1998), Morales (2002), and Borgers, Morales and Sarin (2004) for related analysis in learning models with

Under homogeneity this property drives the whole population to choose optimally: optimal actions become popular and hence more likely to be sampled and chosen (see, e.g., Schlag 1998). Heterogeneity interferes with this implicit popularity weighting, leading to the inefficiencies discussed below.[4]

In our model learning is always incomplete: either (i) all individuals converge to make the same choice, which is optimal for one type but suboptimal for the other, or (ii) only a fraction of the populations of both types choose their optimal action. We refer to these two cases as *uniform adoption* and *dual incomplete learning*, respectively. Which one occurs is determined by two factors: (i) each type's *size*, i.e., the fraction of each type in the population and (ii) the *sensitivity* of each type's decision problem, i.e., each type's difference in expected payoffs across actions. Our main results (Lemma 1 and Theorem 1) characterize the long run inefficiency that arises (uniform adoption or dual incomplete learning) in terms of the sizes and sensitivities of each type. If one type is much larger or has a much more sensitive decision problem than the other, the whole population converges to choose that type's optimal action. On the other hand, a population that is relatively balanced in terms of sizes and sensitivities exhibits dual incomplete learning.

In our benchmark model, we assume *uniform sampling*, i.e., each individual samples each of the others with uniform probabilities. Our framework, however, can be tractably generalized to allow for *biased sampling*. We consider both *homophily* and *heterophily*, i.e., bias toward sampling individuals of the same type and of a different type, respectively.[5] We show (Proposition 2) that the predictions are qualitatively similar to those obtained under uniform sampling. Furthermore, the more homophilous a type is, the greater their fraction that choose their optimal action and the smaller is the fraction of the other type that ends up choosing their optimal action.

We apply our results to a model where a rational monopolist, who has structural knowledge of the economy, faces boundedly rational consumers with heterogeneous valuations for her good.[6] The population is composed of two types of consumers: those whose expected valuation of the good is positive and those whose expected valuation is zero. We characterize the profit-maximizing price of the monopolist and show that bounded rationality leads to lower profits for the monopoly and a larger deadweight loss in comparison to full information. The monopolist cannot charge a price that extracts all the consumer surplus from consumers with positive expected valuation. This would make their problem sensitivity equal to zero and all consumers would converge

linear response to observed payoffs.

[4]The inefficiencies caused by heterogeneity are not confined to payoff-ordering decision rules. Indeed, it can be shown that a decision rule would lead both populations to converge to choose their respective optimal actions only if the problem is "trivial", i.e., if for each individual, the revised probability of choosing her optimal action is one every time she observes both actions.

[5]Homophilic tendencies are widely documented (see Currarini, Jackson and Pin 2009 and the references therein). The results of other extensions are discussed in section 6.

[6]Models in which rational firms interact with boundedly rational consumers in the market have recently been studied, for instance, in Spiegler (2006) and (2011) and Alos-Ferrer, Kirchsteiger and Walzl (2010).

to not buying the good. Therefore, consumers with positive expected valuation are better off than under full information. The other consumers, however, converge to buy a good that they do not value and hence, are worse off than under full information.

Related literature. In Ellison and Fudenberg (1993, 1995) and Schlag (1998), individuals receive feedback about their own and others' choices and make decisions according to cognitively simple rules (for a thorough survey, see Alos-Ferrer and Schlag 2009). In homogeneous populations, relatively simple imitation rules allow most individuals to choose the optimal action over the long run (e.g., Ellison and Fudenberg 1995, Schlag 1998). In contrast, in our analysis, the uniform adoption of one action is not optimal. Furthermore, it is possible that both actions are chosen over the long run by a positive fraction of each type.[7]

In Ellison and Fudenberg (1993), individuals observe only neighbors who, in most cases, have the same optimal actions. The smaller the "window" of neighbors an individual observes, the greater is the fraction of the population that chooses an optimal action in the long run. This is somewhat related to our result that more segregated types achieve better long-run outcomes. Their assumptions on the structure of sampling and heterogeneity, however, prevent their model from generating uniform adoption[8] or allowing the sizes of the types to play a role in the analysis.[9]

Finally, Neary (2012) analyzes a population of individuals who repeatedly play an asymmetric coordination game. He finds that an action is more likely to appear in the long run if it is preferred by the largest type, or if it is more strongly preferred by one of the types. While Neary's (2012) focus is on Group-Darwinian dynamics within strategic settings, our analysis aims to reveal the merits and limitations of simple imitation rules in problems of decision under uncertainty.

2 Framework

In this section we provide the framework from which the parameters of the dynamical system (to be formally introduced in the next section) are derived.

Individual-action types. There are two types of individuals in the population W, denoted by A and B, i.e., $W = A \cup B$. The measure of the set of type A individuals in the population is denoted by $\alpha \in (0, 1)$ and the measure of type B individuals is $1 - \alpha$; furthermore, $\tau, \tau' \in T := \{A, B\}$ denote generic types of

[7] Along with this literature, our analysis is fundamentally different from the study of Bayesian sequential observational learning (e.g., Banerjee 1992, Bikhchandani, Hirshleifer and Welch 1992, and Smith and Sorensen 2000).

[8] Since our benchmark model assumes uniform sampling and generates uniform adoption, and Ellison and Fudenberg (1993) assume non-uniform sampling and do not generate uniform adoption, one may ask whether uniform sampling is what leads to uniform adoption (when it occurs). Our extension to biased sampling, however, reveals that in our model uniform adoption is consistent with non-uniform sampling.

[9] An additional technical difference is that, while in Ellison and Fudenberg (1993, 1995) and Schlag (1998) time is discrete, here we have chosen to analyze a continuous time model (for details, see Section 3).

individuals. At each time $t \in \mathbb{R}^+$, each individual $i \in W$ has chosen an action $c \in S := \{a, b\}$ and she revises this decision from time to time as described below.

Payoffs and comparison signals. The chosen action yields a payoff (rate) x in a finite set $X \subset [0, 1]$; thus, there is a lower and upper bound for payoffs represented by 0 and 1, respectively. The payoff distribution is the same for all type τ individuals, it is time homogeneous, and its expected value and distribution function are denoted by $\pi_{\tau c}$ and $\mu_{\tau c}$, respectively, for all $\tau \in T$ and $c \in S$. In particular, an individual's payoff at time t depends only on her choice and type (and the state of the world) –but not on the choices of other individuals. Individuals do not know $\pi_{\tau c}$ or $\mu_{\tau c}$. We assume that a is the optimal action for type A individuals and b is the optimal action for type B, i.e., $\pi_{Aa} > \pi_{Ab}$ and $\pi_{Ba} < \pi_{Bb}$. We refer to the gains of type A individuals from choosing their optimal action over their suboptimal action, i.e., $\pi_{Aa} - \pi_{Ab}$, as the *sensitivity* of their decision problem. The sensitivity of the decision problem of type B individuals is analogously defined.

When at time t, a type $\tau \in T$ individual i observes a type $\tau' \in T$ individual who has chosen $d \in S$, she also observes a *comparison signal*, denoted by δ. The comparison signal takes values in a finite set $\Delta \subset [-1, 1]$; its distribution function, denoted by $\mu_{\tau\tau' d}$, is assumed to be time homogeneous and the same for all type τ individuals observing a type τ' individual who has chosen d. The comparison signal is interpreted as information about the relative performance of the type τ individual if she were to choose d, compared to the performance of a type τ' individual when he chooses d. The comparison signal is assumed to be *unbiased* in the sense that its expected value, denoted by $\pi_{\tau\tau' d}$, is assumed to be equal to $\pi_{\tau d} - \pi_{\tau' d}$. Therefore, the expected value of the comparison signal is positive (negative) when the individual who observes it would do better (worse) with d than the observed individual. For instance, if i observes $\delta > 0$ when she samples j who chose d, this may be interpreted as the judgement "on average, I would obtain δ more than j if I choose d."

Individual states. We define *individual states* as a combination of type, action, obtained payoff, and comparison signals $\sigma := (\tau, c, x, \delta_A, \delta_B) \in T \times S \times X \times \Delta^2 =: \Sigma$. The comparison signals δ_A and δ_B act as latent variables such that at most one of them activates when a different action is observed to be chosen by a type A individual or a type B individual, respectively, the next time that the individual revises her action.

Intensities and sampling. At time $t \in \mathbb{R}^+$, every individual observes another individual whose state is $\sigma \in \Sigma$ with an intensity that, under the assumption of uniform sampling adopted here, corresponds exactly to the fraction of individuals in the state σ within the population at time t. Upon observing the action and payoff of the sampled individual and the comparison signal corresponding to the sampled individual's type, the individual reviews her choice using a decision rule that we now describe.

Decision rules. Individuals are boundedly rational and make choices according to a *decision rule*. As is common in the literature (e.g., Ellison and Fudenberg 1995, Cubitt and Sugden 1998), we assume that individuals contemplate switching actions only when they observe an action different from the one that they are currently choosing. In this case, the probability of switching to the sampled action is determined by the decision rule (and otherwise individuals simply stick with their current action). Formally, the decision rule is a function that maps the observed payoffs and the comparison signal to the probability of switching to the other action when it is sampled. We denote this function by L, thus $L : [0,1]^2 \times [-1,1] \to [0,1]$. Therefore, $L(x, y, \delta)$ is the probability that an individual switches, given that she obtained the payoff x, observed an individual who chose a different action and obtained the payoff y, and observed the comparison signal δ. Notice that the decision rule L is valid only when a different action is observed, and that $\delta = \delta_A$ if the sampled individual's type is A and $\delta = \delta_B$ if the sampled individual's type is B.[10]

In spite of their severe information restrictions, there are decision rules that allow individuals to be more likely to choose their optimal action every time they observe two different actions (and hence contemplate switching) –regardless of the specific payoff distributions or those of the comparison signals. We call this property *payoff-ordering* and show that only decision rules that are linear in observed payoffs and the comparison signal satisfy this property. Below, in Section 3, we assume that payoffs and comparison signals are independent from each other and across individuals, but for now we aim to define payoff-ordering in a more robust manner, without imposing independence. In particular we assume that the probability mass function of the join vector $(x, y, \delta) \in X \times X \times \Delta$ is time homogenous and the same for every pair of individuals $(i, j) \in W^2$ whose respective types and actions are (τ, c) and (τ', d), with $c \neq d$. The corresponding probability mass function is denoted by $\mu_{\tau c,\tau' d}$ and the expected value of L is denoted by $L_{cd}(\tau, \tau') := \sum_{(x,y,\delta)\in X\times X\times\Delta} \mu_{\tau c,\tau' d}(x, y, \delta)L(x, y, \delta)$. I.e., $L_{cd}(\tau, \tau')$ is the expected value of the probability that a type τ individual who chooses c and observes a type τ' individual who chooses d, switches from c to d.

Definition 1 *A decision rule L is payoff-ordering if for any two different actions c and d, if $\pi_{\tau d} > (<)\, \pi_{\tau c}$, then $L_{cd}(\tau, \tau') > (<)\, \frac{1}{2}$ for all $\tau, \tau' \in T$ and probability mass function $\mu_{\tau c,\tau' d} : X \times X \times \Delta \to [0,1]$, with arbitrary finite sets $X \subset [0,1]$ and $\Delta \subset [-1,1]$.*[11]

Proposition 1 *L is payoff-ordering if and only if $L(x, y, \delta) = \frac{1}{2} + \beta(y + \delta - x)$, with $\beta \in (0, 1/4]$ for all $x, y \in [0,1]$, and $\delta \in [-1,1]$.*[12]

[10]Given that we introduce comparison signals, calling our decision rule a model of imitation might be a bit controversial. We follow this convention because the most related models in the literature are usually called imitation models (see, e.g., Alos-Ferrer and Schlag 2009).

[11]The assumption that the comparison signal is unbiased is still imposed here and throughout the paper, except in Appendix F, where we study the effect of abandoning it.

[12]Since $y + \delta - x \in [-2, 2]$, we need $\beta \in (0, 1/4]$ so that the probabilities of switching are specified in $[0,1]$. In general, when $(y + \delta - x) \in [-\bar{c}, \bar{c}]$, with $\bar{c} > 0$, we need $\beta \in \left(0, \frac{1}{2\bar{c}}\right]$. Here $\bar{c} = 2$, but, for instance, the assumptions we make in the application of Section 5 yield a different value for $\bar{c}$.

The proof for the 'if' statement is instructive and is provided here. The argument for the 'only if' part is provided in Appendix A.

Proof. For any $\tau, \tau' \in T$, and $d \neq c \in S$,

$$\begin{aligned} L_{cd}(\tau, \tau') &= \sum_{(x,y,\delta)\in X^2\times\Delta} \mu_{\tau c,\tau' d}(x, y, \delta)\left(\frac{1}{2} + \beta(y + \delta - x)\right) \\ &= \frac{1}{2} + \beta(\pi_{\tau' d} + \pi_{\tau d} - \pi_{\tau' d} - \pi_{\tau c}) \\ &= \frac{1}{2} + \beta(\pi_{\tau d} - \pi_{\tau c}). \end{aligned}$$

Therefore, if $\pi_{\tau d} > \pi_{\tau c}$, then $L_{cd}(\tau, \tau') > 1/2$. ■

From now on, unless stated otherwise, we assume that decision rules are payoff-ordering. Since the comparison signal is unbiased, $y + \delta$ is an unbiased estimator of the expected payoff of individual i if she chooses the action of the sampled individual j. Therefore, $y + \delta - x$ is an unbiased estimator of the difference between the expected payoff of d and c for individual i. Hence, since $\beta > 0$, in expected value, the probability of choosing the action that provides her the greatest expected payoff is greater than the probability of choosing the action that provides her the smallest expected payoff. Notice also that since $\beta \in \left(0, \frac{1}{4}\right]$ and $\pi_{\tau d} - \pi_{\tau c} \leq 1$, by the proof of Proposition 1, $L_{cd}(\tau, \tau') \in \left(\frac{1}{2}, \frac{3}{4}\right]$ when $\pi_{\tau d} > \pi_{\tau c}$ and $L_{cd}(\tau, \tau') \in \left[\frac{1}{4}, \frac{1}{2}\right]$ otherwise.

Previous work on imitation in homogeneous populations focuses on either exogenously given rules or decision rules that have been shown to satisfy some desirable properties. For instance, Schlag (1998) studies *improving* rules, which satisfy that the population's average expected payoff is expected to be non-decreasing in time. It is easy to show that in our model, heterogeneity rules out this possibility. It would also be desirable for the decision rule to guarantee that individuals not only would be more likely to choose their optimal action, but also that they do so with high probability. Unfortunately, this is not possible in our model. Indeed, from the proof of Proposition 1 we have that for every decision rule, the expected value of the updated probability of choosing the action with the highest payoff of two observed actions is arbitrarily close to one half for some probability mass functions $\mu_{\tau c,\tau' d}$. The expected probability of choosing the optimal action for any environment depends on the value of β. This probability is maximized at $\beta = 1/4$, yet the subsequent analysis is valid for all $\beta \in (0, 1/4]$.[13]

[13] It may seem unintuitive that a payoff-ordering decision rule prescribes switching with positive probability even if the observed action earned a lower perceived payoff than the own action. Changing this feature of the decision rule, however, would not allow payoff-ordering as the decision would necessarily have to be non-linear in observed payoffs and then the construction in the argument of sufficiency of the proof of Proposition 1 would not go through. To illustrate, consider an environment in which c gives a payoff of 0.1 to type A individuals with certainty, d gives a payoff of 0 with certainty for type B individuals and the comparison signal observed by type A individuals observing type B individuals choosing d is equal to 0 with probability 4/5 and 0.9 with probability 1/5. In this case $\pi_{Ad} > \pi_{Ac}$. Nevertheless, a decision rule that never switches if the observed action's perceived payoff is lower than the own would require a type A individual to stick with c with probability 4/5 when observing a type B individual

Finally, we notice that observing comparison signals allows individuals to make choices that, in expected value, do not depend on the observed individual's type and depend only on his choice. This result follows from the assumption that the comparison signal is unbiased; the formal argument for its proof follows directly from the proof of sufficiency in Proposition 1, and it is omitted.

Remark 1 *If L is payoff-ordering, then $L_{cd}(\tau,\tau) = L_{cd}(\tau,\tau')$ for all $\tau,\tau' \in T$ and different actions $c,d \in S$.*

Hence, for any payoff-ordering decision rule we define $L_{cd}(\tau) := L_{cd}(\tau,\tau')$ for any different actions $c,d \in S$ and $\tau,\tau' \in T$.

Individual state update. As mentioned above, each individual reviews her choice at points in time determined by the intensities, i.e., the probability of sampling an individual of each state $\sigma \in \Sigma$.[14] We assume that, upon sampling, an individual whose state is $\sigma = (\tau, c, x, \delta_A, \delta_B)$ and samples an individual with state $\sigma' = (\tau', c', x', \delta'_A, \delta'_B)$ updates her state to $\sigma'' = (\tau'', c'', x'', \delta''_A, \delta''_B)$ with probability $\varsigma_{\sigma\sigma'}(\sigma'')$, given by

$$\varsigma_{\sigma\sigma'}(\sigma'') = \begin{cases} \mu_{\tau c}(x'')\mu_{\tau Ad}(\delta''_A)\mu_{\tau Bd}(\delta''_B) & \text{if} \quad c = c' = c'' \text{ and } \tau'' = \tau \\ L(x, x', \delta_{\tau'})\mu_{\tau c'}(x'')\mu_{\tau Ac}(\delta''_A)\mu_{\tau Bc}(\delta''_B) & \text{if} \quad c'' = c' \neq c \text{ and } \tau'' = \tau \\ (1 - L(x, x', \delta_{\tau'}))\, \mu_{\tau c}(x'')\mu_{\tau Ac'}(\delta''_A)\mu_{\tau Bc'}(\delta''_B) & \text{if} \quad c'' = c \neq c' \text{ and } \tau'' = \tau \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $d \neq c$. This assumption effectively imposes independence between an individual's payoff and the comparison signal she observes the next time she reviews her action.

3 Population dynamics

In this section we analyze the dynamics of choices when individuals choose according to a payoff-ordering decision rule. The aim is to use a continuum population as an approximation of what would happen in an economy populated by such a large population that the effect of idiosyncratic shocks vanishes by virtue of the Law of Large numbers. In particular, we apply results in Duffie, Qiao and Sun (2016) (see also Sun 2006 and Duffie and Sun 2007 and 2012) to construct a dynamical system in which an Exact Law of Large numbers can be applied so that the fractions of the populations of each type choosing each action are those corresponding to the dynamics of their expected values. The formal derivation of this dynamical system is provided in

who chose d.

[14] Similarly, the standard approach in the literature assumes that action revision times are determined according to an independent Poisson distribution. See, e.g., Sandholm (2010a, 2010b), and Hofbauer and Sandholm (2011) for further details.

Appendix B.[15]

The model proceeds in continuous time. Each individual reviews her choice at points in time according to the intensities and sampling procedures described in Section 2. Upon sampling, individuals update their state according to (1). Our analysis is concerned with the fraction of type A individuals choosing a and the fraction of type B individuals choosing b. These fractions follow a stochastic process fully determined by the sampling and updating procedures described above. Since we assume a continuum population, our analysis focuses on the deterministic path corresponding to the expected value of the stochastic process. Let $p(t)$ be the fraction of type A individuals choosing a and let $q(t)$ be the fraction of type B individuals choosing b, for all $t \geq 0$, along this path. If the fraction of individuals whose state is σ at time t is denoted by $p_\sigma(t)$ for all $\sigma \in \Sigma$ and $t \in \mathbb{R}^+$, we have that

$$\alpha p(t) = \sum_{\sigma \in \Sigma : \tau = A, c = a} p_\sigma(t) \tag{2}$$

and

$$(1-\alpha) q(t) = \sum_{\sigma \in \Sigma : \tau = B, c = b} p_\sigma(t). \tag{3}$$

Let $\mathit{\Delta}(\Sigma)$ be the set of all possible distributions of individual states at any point in time; that is,

$$\mathit{\Delta}(\Sigma) := \left\{ p \in \mathbb{R}^{|\Sigma|} : p_k \geq 0 \text{ for } k = 1, 2, ..., |\Sigma| \text{ and } \sum_{k=1}^{|\Sigma|} p_k = 1 \right\}.$$

The initial fractions of the population in each individual state, denoted by $(p_\sigma(0))_{\sigma \in \Sigma} \in \mathit{\Delta}(\Sigma)$, are assumed exogenously given. Yet, these initial conditions need to satisfy that the fractions of the population of type A and B individuals are α and $1-\alpha$, respectively; i.e., $\sum_{\sigma \in \Sigma : \tau = A} p_\sigma(0) = \alpha$ and $\sum_{\sigma \in \Sigma : \tau = B} p_\sigma(0) = 1-\alpha$. Similarly, the initial fraction of type τ individuals who choose action c and obtain a payoff x (among all type τ individuals who choose c) is assumed to be $\mu_{\tau c}(x)$, for all $\tau \in T$, $c \in S$, and $x \in X$; and the fraction of type τ individuals who choose action c, sample a type τ' individual who chooses $d \neq c$, and observe a comparison signal $\delta_{\tau'}$ (among all type τ individuals who choose c and sample a type τ' individual who chooses $d \neq c$) is $\mu_{\tau\tau' d}(\delta_{\tau'})$, for all $\tau \in T$, $\tau' \in T$, $c \in S$, and $\delta_{\tau'} \in \Delta$. The subset of $\mathit{\Delta}(\Sigma)$ that satisfies all these restrictions is

[15] There are measurability problems in invoking the law of large numbers for a continuum of random variables (see, e.g., Feldman and Gilles 1985, Judd 1985, and Alos-Ferrer 2002). We thank Carlos Alos-Ferrer for pointing this out and providing us with these references. In the last decade, a number of papers deal with this issue and with independent random matching in particular, using Fubini Extensions introduced in Sun (2006) (see, e.g., Duffie and Sun 2007, 2012). See also Podczeck and Puzzello (2012), who build on an earlier contribution by Alos-Ferrer (1999) to study independent random matching with a continuum of agents. We thank three anonymous referees for guiding us to this literature. In particular, as mentioned above, the foundations for the dynamical system analyzed here are an application of the results of Duffie, Qiao and Sun (2016). In a related environment, with a different approach, Benaim and Weibull (2003) show that the deterministic path of dynamical systems yield a reasonable approximation of discrete time stochastic adjustment in large populations and over finite time horizons (see also Sandholm 2003 and Sandholm 2010a).

denoted by $\hat{\Delta}(\Sigma)$.

Fix the initial fractions of the population in each individual state at $(p_\sigma(0))_{\sigma\in\Sigma} \in \hat{\Delta}(\Sigma)$. Then, the paths $p(t)$ and $q(t)$, are given by the solution of the system of differential equations

$$\begin{aligned} \alpha\dot{p} &= \alpha(1-p)\left[\alpha p+(1-\alpha)(1-q)\right]L_{ba}(A)-\alpha p\left[\alpha(1-p)+(1-\alpha)q\right]L_{ab}(A) && (4)\\ (1-\alpha)\dot{q} &= (1-\alpha)(1-q)\left[(1-\alpha)q+\alpha(1-p)\right]L_{ab}(B)-(1-\alpha)q\left[(1-\alpha)(1-q)+\alpha p\right]L_{ba}(B), && (5)\end{aligned}$$

with initial condition $(p(0),q(0)) = \left(\alpha^{-1}\sum_{\sigma\in\Sigma:\tau=A,c=a} p_\sigma(0), (1-\alpha)^{-1}\sum_{\sigma\in\Sigma:\tau=B,c=b} p_\sigma(0)\right)$.[16]

The first term on the right-hand side of (4) gives the flow to action a of type A individuals. The mass of type A individuals choosing b and sampling someone choosing a is given by $\alpha(1-p)\left[\alpha p+(1-\alpha)(1-q)\right]$ and the rate at which these individuals switch is given by $L_{ba}(A)$. Similarly, the second term on the right-hand side of (4) gives the flow to action b in the type A population. An analogous interpretation applies to (5).

We define $p,q:\mathbb{R}^+\to[0,1]$ as the solutions of (4)-(5) (with exogenously given initial conditions $(p_\sigma(0))_{\sigma\in\Sigma} \in \hat{\Delta}(\Sigma)$); most of the times, though, the time dependence of $p(t)$ and $q(t)$ will be omitted. Let U and D be the expected value of the probability of switching to their optimal action for type A and type B individuals, respectively, i.e., $U := L_{ba}(A)$ and $D := L_{ab}(B)$. From the proof of sufficiency of Proposition 1, since the difference of expected payoffs across actions is contained in $[-1,1]$, we have $U, D \in (1/2, 3/4]$.[17] Payoff-ordering decision rules satisfy $L_{ab}(\tau) = 1 - L_{ba}(\tau)$ for $\tau \in \{A, B\}$, therefore the system of differential equations (4)-(5) can be written as

$$\begin{aligned} \dot{p} &= \alpha p(1-p)(2U-1)+(1-\alpha)\left[(1-p)(1-q)U-pq(1-U)\right] && (6)\\ \dot{q} &= (1-\alpha)q(1-q)(2D-1)+\alpha\left[(1-q)(1-p)D-qp(1-D)\right]. && (7)\end{aligned}$$

In the sequel, it will often be convenient to work with the functions $\dot{p},\dot{q}:[0,1]^2\to\mathbb{R}$, with $\dot{p}(p,q)$ and $\dot{q}(p,q)$ defined by the right hand sides of (6) and (7), respectively. Notice that $\dot{p}(p,q)$ is decreasing in q. As q increases there are fewer type B individuals choosing a, therefore, when a type A individual samples a type B individual, the probability that this individual has chosen b is greater. This makes it more likely for type A individuals to choose b. The effect of p on $\dot{p}$ is ambiguous. In both heterogeneous and homogeneous populations, $\dot{p}$ is a concave polynomial in p. In a homogeneous population, when p is very small there are too few type

[16] As in most of the equations below, the time dependence of p and q have been omitted.

[17] It may be thought that U and D being bounded away from 1 is a restrictive feature of the model. There is at least a couple of alternative ways to avoid this. First, the distribution of the comparison signal could be allowed to depend on the payoff realization in such a manner that $y+\delta-x\in[-1,1]$. This would allow $U,D\in(1/2,1]$. Alternatively, as we discuss below, one could consider decision rules which are not payoff-ordering. The results of allowing $U,D\in(1/2,1]$, however, are qualitatively similar to those of our benchmark case.

A individuals from whom to sample action a. On the other hand, when p is very large, just a few type A individuals are left to switch from b to a. In a homogeneous population, as long as $p \in (0,1)$, $\dot{p} > 0$. This follows from the fact that, since $U > 1/2$, the flows from a to b are more than compensated by flows in the opposite direction. In a heterogeneous (non-isolated) population, however, $p \in (0,1)$ is compatible with $\dot{p} < 0$. This occurs for high values of p at which an important fraction of type A individuals may be mislead when sampling type B individuals choosing b which may cause $\dot{p} < 0$. An analogous reasoning applies to $\dot{q}$ and (7).

Long run results. Let the set of rest points of the system (6)-(7) be denoted by

$$RP := \{(p,q) \in [0,1]^2 : \dot{p}(p,q) = \dot{q}(p,q) = 0\}.$$

First we characterize RP. When all individuals choose the same action the system is in a rest point. Hence, $(0,1)$ and $(1,0)$ are rest points and we refer to them as *corner rest points*. Our next result shows that for some values of α, U, and D there is a third rest point located in $(0,1)^2$. This *interior rest point* is given by $(\widehat{p}, \widehat{q})$, with

$$\widehat{p} := \frac{U(\alpha(U+D-1) - (1-U)(2D-1))}{\alpha(2U-1)(U+D-1)}$$

and

$$\widehat{q} := \frac{D((1-\alpha)(U+D-1) - (1-D)(2U-1))}{(1-\alpha)(2D-1)(U+D-1)},$$

and it exists if and only if

$$\underline{\alpha} := \frac{(1-U)(2D-1)}{U+D-1} < \alpha < \frac{U(2D-1)}{U+D-1} =: \overline{\alpha}.$$

Lemma 1 *If $\alpha \in (\underline{\alpha}, \overline{\alpha})$, then $RP = \{(0,1),(1,0),(\widehat{p},\widehat{q})\}$, otherwise $RP = \{(0,1),(1,0)\}$.*

In the proof of this lemma we use the functions $\overline{q} : [0,1] \to \mathbb{R}$, defined by $\dot{p}(p, \overline{q}(p)) = 0$ for all $p \in [0,1]$, and $\overline{p} : [0,1] \to \mathbb{R}$, defined by $\dot{q}(\overline{p}(q), q) = 0$ for all $q \in [0,1]$.

Proof. If $p = 1$, then $\dot{p} = 0$ if and only if $q = 0$. Correspondingly, if $q = 1$, then $\dot{p} = 0$ if and only if $p = 0$. Hence, $(1,0)$ and $(0,1)$ are rest points.

From (6) and (7), the points (p,q) that satisfy $\dot{p} = \dot{q} = 0$ are those that satisfy both

$$q = \frac{\alpha p(1-p)(2U-1) + (1-\alpha)(1-p)U}{(1-\alpha)((1-p)U + p(1-U))}, \tag{8}$$

$$p = \frac{(1-\alpha)q(1-q)(2D-1) + \alpha(1-q)D}{\alpha((1-q)D + q(1-D))}. \tag{9}$$

Notice that $\overline{q}(p)$ and $\overline{p}(q)$ are given by the right-hand side of (8) and (9), respectively, and hence, their graphs contain $(0,1)$ and $(1,0)$. The second derivative of $\overline{q}(p)$ is

$$\overline{q}''(p) \quad = \quad \frac{2U(2U-1)(1-U)}{(1-\alpha)(p(2U-1)-U)^3},$$

where the numerator is strictly positive and the denominator is strictly negative, because $p \leq 1 < U/(2U-1)$; hence $\overline{q}$ is strictly concave. A similar calculation reveals that $\overline{p}$ is strictly concave as well. If (p, q) is a rest point of the system, then $p = \overline{p}(\overline{q}(p))$ and $q = \overline{q}(p)$ (or equivalently, $q = \overline{q}(\overline{p}(q))$ and $p = \overline{p}(q)$). Hence, for (p, q) to be a rest point, p must satisfy

$$\overline{p}(\overline{q}(p)) = \frac{(1-\alpha)\overline{q}(p)(1-\overline{q}(p))(2D-1) + \alpha(1-\overline{q}(p))D}{\alpha((1-\overline{q}(p))D + \overline{q}(p)(1-D))} = p.$$

This yields $p = \widehat{p}$. For this to be a rest point not in $\{(0,1),(1,0)\}$ we also need $\widehat{p} \in (0,1)$. The inequality $\widehat{p} > 0$ simplifies to $\alpha > \underline{\alpha}$ and $\widehat{p} < 1$ simplifies to $\alpha < \overline{\alpha}$, i.e., $\widehat{p} \in (0,1)$ if and only if $\alpha \in (\underline{\alpha}, \overline{\alpha})$. Further, $\overline{q}(\widehat{p}) = \widehat{q}$ and $\widehat{q} \in (0,1)$ if and only if $\alpha \in (\underline{\alpha}, \overline{\alpha})$. Hence, there is an interior rest point given by $(\widehat{p}, \widehat{q})$ if and only if $\alpha \in (\underline{\alpha}, \overline{\alpha})$. ■

We refer to corner rest points as *uniform adoption* and interior rest points as *dual incomplete learning*. Next we analyze the conditions for the different rest points to be stable. The notion of asymptotic stability that we study requires the system to remain close and converge to the rest point whenever the system starts sufficiently close to it (e.g., Hofbauer and Sigmund 1998). Formally a rest point (p^*, q^*) is *asymptotically stable* if (i) for any $\varepsilon > 0$ there exists some $\gamma_1 \in (0, \varepsilon)$ such that if $||(p(t), q(t)) - (p^*, q^*)|| < \gamma_1$, then $||(p(t'), q(t')) - (p^*, q^*)|| < \varepsilon$ for all $t' > t$, and (ii) there exists some $\gamma_2 > 0$ such that if $||(p(t), q(t)) - (p^*, q^*)|| < \gamma_2$, then $\lim_{t' \to \infty}(p(t'), q(t')) = (p^*, q^*)$.[18]

In the following theorem, we characterize the stability properties of the different rest points of system (4)-(5) for virtually all possible parameter values.[19] Furthermore, we show that in each case, the asymptotically stable rest point is a *global attractor*, i.e., the system converges to this point regardless of the initial conditions (as long as the path does not start at a different rest point). The theorem shows that, if the fraction of type A individuals or the fraction of type B individuals is small enough, i.e., if $\alpha < \underline{\alpha}$ or $\alpha > \overline{\alpha}$, then we have convergence to uniform adoption. Otherwise, we have convergence to dual incomplete learning. As explained above, the cutoffs $\underline{\alpha}$ and $\overline{\alpha}$ are determined by the sensitivities of the decision problems of each type.[20]

[18] $|| \cdot ||$ stands for the Euclidean norm.

[19] We only exclude $\alpha = \underline{\alpha}$ and $\alpha = \overline{\alpha}$, because the techniques that we use in the proof of the result applies only to hyperbolic rest points (see the proof of Theorem 1) and at $\alpha = \underline{\alpha}$ and $\alpha = \overline{\alpha}$ some of the rest points are not hyperbolic.

[20] We cannot have complete learning for one type and partial learning for the other. E.g., we have no rest points $(1, q)$ with $q > 0$. Intuitively, since type A individuals occasionally observe action b in this point and switch with positive probability when they do, they would move away from action a. Similarly, we have no rest points $(p, 1)$ with $p > 0$.

Theorem 1 *Suppose $\alpha \neq \underline{\alpha}$ and $\alpha \neq \overline{\alpha}$. Then, the system (6)-(7) has a unique asymptotically stable point given by*

$$(p^*, q^*) = \begin{cases} (0,1) & \text{if} \quad \alpha < \underline{\alpha} \\ (\hat{p}, \hat{q}) & \text{if} \quad \underline{\alpha} < \alpha < \overline{\alpha} \\ (1,0) & \text{if} \quad \alpha > \overline{\alpha}. \end{cases}$$

Furthermore, in each of these cases $\lim_{t\to\infty}(p(t), q(t)) = (p^*, q^*)$ *for all paths such that* $(p(0), q(0)) \notin RP \setminus \{(p^*, q^*)\}$.[21]

The proof of Theorem 1 is provided in Appendix C. Intuitively, since individuals make decisions according to a payoff ordering decision rule, in each decision they are more likely to choose their optimal action. The smaller the fraction of type A individuals in the population, however, the less likely it is to sample an individual choosing a. This leads action a to propagate less and eventually the whole population converges to choose b. Analogously, when the fraction of type A individuals in the population is large enough, a propagates more than b, and eventually, the whole population ends up choosing a. Finally, if the fraction of the population of type A is neither large enough nor small enough, both actions propagate in a more balanced manner and the system converges to an interior asymptotically stable rest point.

The fact that we do not have efficiency in the long run, i.e., that (p, q) does not converge to $(1, 1)$ is a robust phenomenon that does not rely on the linearity of payoff-ordering decision rules. To see this, notice that for an arbitrary decision rule L, system (4)-(5) at $(p, q) = (1, 1)$ yields $\dot{p} = -(1-\alpha)L_{ab}(A, B)$ and $\dot{q} = -\alpha L_{ba}(B, A)$. Hence, unless switching to the sub-optimal action occurs with probability zero for both types, i.e., $L_{ab}(A, B) = L_{ba}(B, A) = 0$, the system cannot converge to $(1, 1)$ if it starts elsewhere. The conditions $L_{ab}(A, B) = L_{ba}(B, A) = 0$ are fairly restrictive, however, as they require the decision maker never to switch in any event of realizations of observed payoffs and comparison signal, when she chose her optimal action.[22]

Dynamics of the system. We illustrate the dynamics of the system in the phase diagrams displayed in Figure 1. The isoclines correspond to the graphs of the functions $\overline{p}$ and $\overline{q}$. The left, center and right panels correspond to the cases $\alpha < \underline{\alpha}$, $\underline{\alpha} < \alpha < \overline{\alpha}$, and $\overline{\alpha} < \alpha$, respectively. As illustrated in Figure 1, the distinctive feature of each of these cases is that $\overline{p}$ is above $\overline{q}$, $\overline{p}$ and $\overline{q}$ intersect, and $\overline{p}$ is below $\overline{q}$, respectively.

[21] Whenever a rest point is a global attractor, it satisfies the second condition of asymptotic stability. In general, however, it does not necessarily satisfy the first part, i.e., it is possible that the dynamics move away from the rest point before eventually converging to it. Nevertheless, Theorem 1 reveals that a rest point of (6)-(7) is asymptotically stable if and only if it is a global attractor of the system.

[22] There are, however, examples of decision rules and environments such that these conditions are met. For instance, consider decision rules such as a suitably modified version of "imitate if better" (see, e.g., Oyarzun and Ruf 2009), in which individuals switch (corresp. do not switch) with probability 1 if $y + \delta > x$ (corresp. $y + \delta < x$), and randomize with a fair coin, otherwise. This decision rule yields $L_{ab}(A, B) = L_{ba}(B, A) = 0$ in the environment such that $\pi_{Aa} = \pi_{Bb} = 1$, $\pi_{Ab} = \pi_{Ba} = 0$, and the comparison signals within individuals of the same type (for all actions) are deterministic.

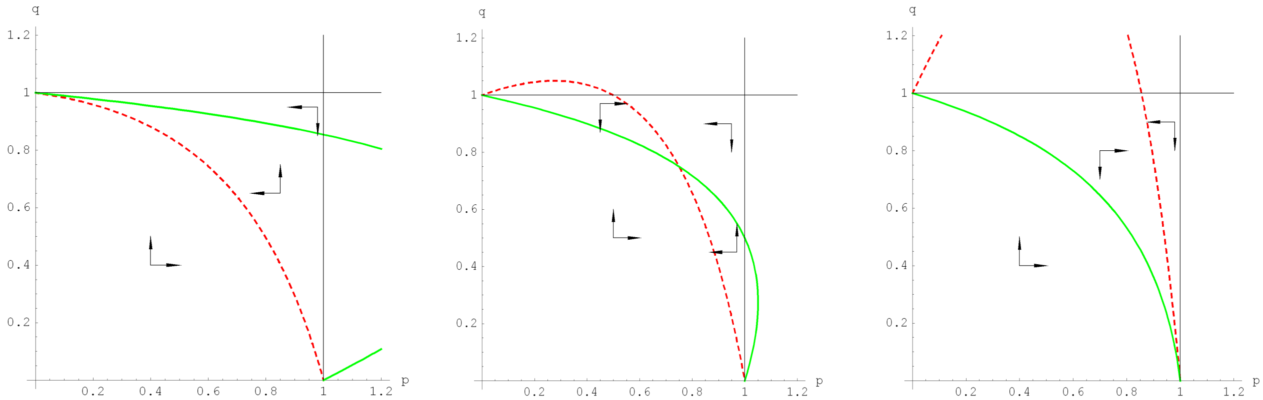

Figure 1. The solid and dashed lines represent $\overline{p}$ and $\overline{q}$, respectively. $U = D = 3/4$ ($\underline{\alpha}= 1/4$ and $\overline{\alpha}= 3/4$) and $\alpha= 9/40$, $1/2$, and $31/40$ in the left, center, and right panel, respectively.

Let us provide some intuition for why we obtain such patterns in the dynamics of the system. For concreteness, we focus on the case in which both $p(0)$ and $q(0)$ are small and the population converge to choose b (the left panel of Figure 1). When both p and q are initially small, there is a large amount of individuals in each type that are subject to switch to their respective optimal action. Further, whenever an individual samples an individual from the opposite type she is likely to sample her own optimal action. Thus, initially learning occurs for both types of individuals in the sense that both p and q increase. In a homogeneous population imitation would continue and the whole population would converge to choose the unique optimal action. In our model, however, (p, q) eventually reaches a point above the isocline $\dot{p} = 0$ and when this happens p starts to decrease. At this point the measure of type A individuals who are subject to switch to a and the measure of type B individuals from whom a can be sampled are both smaller than at the beginning, reversing the increase in p. The decrease in p benefits type B, who become more likely to sample b from type A. This sets q on a positive trend, which in turn accelerates the decrease in p. The result is a dynamic in which the fraction of type A individuals choosing a decreases until it converges to zero, while the fraction of type B individuals choosing b converges monotonically to one. The analysis of the other cases is similar and is left for the reader.

The model predicts convergence to a corner rest point for a large set of parameter values. Therefore, in our model, when the sensitivities of the problems are relatively similar, imitation within a heterogeneous population may not result in optimality for minorities. This contrasts sharply with findings for homogeneous populations, where such adverse effects cannot arise (e.g., Ellison and Fudenberg 1995, Schlag 1998). The adverse effect for the minority is related to the fact that its optimal action eventually becomes difficult to sample. The sampling procedure creates a bias toward actions that are more popular, in the sense that a larger fraction of the population is choosing them. This contrasts with the manner in which Ellison and Fudenberg (1993) introduce popularity, in which they assume that when individuals make choices, they are biased toward more popular actions. In our analysis, there is no exogenously imposed bias toward more popular actions. As in Ellison and Fudenberg (1995), this bias instead arises endogenously as a result of the sampling procedure.

Comparative statics: the role of sensitivity and size. Here we analyze the impact of α, U, and D on the predictions of the model. Recall that U and D are determined by the difference in the expected payoff across actions for type A and type B individuals, respectively. Hence, the effect of U and D on the long run outcomes reflects the impact of the sensitivity of the decision problems. Abusing notation, let the functions $\underline{\alpha} : (1/2, 3/4]^2 \to \mathbb{R}$ and $\overline{\alpha} : (1/2, 3/4]^2 \to \mathbb{R}$ be defined by $\underline{\alpha}(U,D) = \frac{(1-U)(2D-1)}{U+D-1}$ and $\overline{\alpha}(U,D) = \frac{U(2D-1)}{U+D-1}$, respectively, for all $(U,D) \in (1/2, 3/4]^2$. An interior rest point exists whenever $\underline{\alpha}(U,D) < \alpha < \overline{\alpha}(U,D)$. Let $\underline{\alpha}_i$ and $\overline{\alpha}_i$ denote the first derivative of $\underline{\alpha}$ and $\overline{\alpha}$, respectively, with respect to $i = U, D$. We obtain $\underline{\alpha}_U(U,D) < 0$ and $\overline{\alpha}_U(U,D) < 0$. Thus for greater values of U the population converge to choose b for a smaller set of values of α and a for a larger set of values of α. Similarly $\underline{\alpha}_D(U,D) > 0$ and $\overline{\alpha}_D(U,D) > 0$.

Notice also that $\lim_{D\to 1/2} \underline{\alpha}(U,D) = \lim_{D\to 1/2} \overline{\alpha}(U,D) = 0$ and $\lim_{U\to 1/2} \underline{\alpha}(U,D) = \lim_{U\to 1/2} \overline{\alpha}(U,D) = 1$. This implies that for small enough D (and for fixed α and U) the population converges to choose a, whereas for small enough U (and for fixed α and D) the population converges to choose b. Hence, if the majority is close to indifferent between a and b, but the minority is not, the population converges to choose the minority's optimal action. On the other hand, since $\underline{\alpha}(U,D), \overline{\alpha}(U,D) \in (0,1)$, for any U, D there exist some α such that the population converge to choose either a or b. This means that even if, for example, type A individuals are close to indifferent between a and b, if type A is sufficiently large, the population converge to choose a.

Let $(\hat{p}(U,D,\alpha), \hat{q}(U,D,\alpha))$ be the interior rest point expressed as a function of the parameters of the model. Then, for $\alpha \in (\underline{\alpha}(U,D), \overline{\alpha}(U,D))$ we obtain $\widehat{p}_U > 0$, $\widehat{p}_D < 0$, and $\widehat{p}_\alpha > 0$, where $\widehat{p}_i$, denote the first derivative of $\hat{p}(U,D,\alpha)$ with respect to i, for $i = U, D, \alpha$. Hence, the fraction of type A individuals choosing their optimal action in an interior rest point increases in type A's sensitivity and size and decreases in type B's sensitivity. In the interior rest point, $\dot{p}(p,q)$ is decreasing in p. As $\dot{p}(p,q)$ is increasing in U, when U increases, the isocline $\dot{p}(p,q) = 0$ moves to the right which results in higher value of p in equilibrium. A similar argument reveals why $\widehat{p}$ is increasing in α and decreasing in D.

Homogeneous sensitivities. It is instructive to look at the case where the sensitivity of the problems of both types of individuals is the same. In the special case $U = D$ and $\alpha = 1/2$ we obtain $\widehat{p} = \widehat{q} = U$. Let us briefly look at why this is the case. If $p = q = U$, then the probability of sampling either action is equal to $1/2$. The flow towards a among type A individuals is therefore $\frac{1}{2}(1-p)U = \frac{1}{2}(1-U)U$ and the flow towards b in A is $\frac{1}{2}p(1-U) = \frac{1}{2}U(1-U)$. Hence, switching towards and switching away from a among type A individuals are balanced.

More generally, if $U = D$, we have that $\underline{\alpha} = 1 - U$, $\overline{\alpha} = U$, and $\widehat{p} = \frac{U}{2U-1}\frac{U-(1-\alpha)}{\alpha}$ and $\widehat{q} = \frac{U}{2U-1}\frac{U-\alpha}{1-\alpha}$ for all $\alpha \in (1-U, U)$. This implies that $\alpha\widehat{p} = U\frac{U-(1-\alpha)}{2U-1}$ and $(1-\alpha)\widehat{q} = U\frac{U-\alpha}{2U-1}$. Hence, $\alpha\widehat{p} + (1-\alpha)\widehat{q} = U$, so the total fraction of individuals in the population choosing their optimal action in the interior rest point is equal to the common sensitivity of both types. In the corner equilibria, however, when $\alpha \notin (1-U, U)$, the fraction of the population choosing their optimal action is greater than U.

Non payoff-ordering decision rules. We now turn our attention to the impact of assuming that individuals follow payoff-ordering decision rules. Alternatively, we could consider decision rules such as the version of "imitate if better" described in footnote 22. It is easy to construct environments such that $U, D \in (3/4, 1)$ for this decision rule. This yields results qualitatively similar to those provided above. More possibilities arise when U and D can be less than $1/2$, which can also occur in some environments for this decision rule. If $U \leq 1/2$ and $D > 1/2$, then the population converge to choose b for all $\alpha \in (0,1)$, and hence, as with payoff-ordering decision rules, we obtain uniform adoption. A different result, however, can be obtained if both $U, D < 1/2$. Then, for a range of values of α, an interior equilibrium may exist in which both $p^*, q^* < 1/2$. A decision rule yielding $U, D < 1/2$, however, would certainly be very unappealing. In summary, assuming payoff-ordering we rule out $U, D < 1/2$ and thus, that both types of individuals do worse in the long run than with simple random choice.

Average expected payoff of the population. If individuals were to randomize uniformly between the two actions, then, on average, half of the individuals of each type would choose their optimal action. We say that imitation is *detrimental* for a type of individuals whenever less than half of them choose their optimal action asymptotically. The imitation process is detrimental for one of the types when the asymptotically stable rest point is either $(0,1)$ or $(1,0)$. At an interior rest point, imitation is detrimental for at most one type: since $\overline{q}(0) = 1$ and $\overline{q}(1) = 0$, and $\overline{q}(p)$ is strictly concave, we have $\overline{q}(p) > 1 - p$ for all $p \in (0,1)$; therefore, $\hat{p} + \hat{q} > 1$. If there is a type for which imitation is detrimental, however, this type represents a small fraction of the population or has the decision problem with the smallest sensitivity. It turns out that the gains over random choice of the type that benefits from imitation always exceed the losses incurred by the type for whom imitation is detrimental, if any. The average expected payoff of the population in the state (p,q), denoted by $W^I(p,q)$, is given by

$$W^I(p,q) = \alpha(p\pi_{Aa} + (1-p)\pi_{Ab}) + (1-\alpha)(q\pi_{Bb} + (1-q)\pi_{Ba}).$$

The average expected payoff of the population when all individuals choose randomly with uniform probability, denoted by W^{RC}, is given by

$$W^{RC} = \frac{1}{2}(\alpha(\pi_{Aa} + \pi_{Ab}) + (1-\alpha)(\pi_{Bb} + \pi_{Ba})).$$

Remark 2 *$W^I(p^*, q^*) > W^{RC}$ for any asymptotically stable rest point (p^*, q^*) of (4)-(5).*

Proof. First, consider $\alpha > \overline{\alpha}$. Then $(p^*, q^*) = (1, 0)$ and

$$W^I(1,0) - W^{RC} = \frac{1}{2}(\alpha(\pi_{Aa} - \pi_{Ab}) - (1-\alpha)(\pi_{Bb} - \pi_{Ba})) = \frac{1}{4\beta}(\alpha(2U-1) - (1-\alpha)(2D-1)),$$

which implies that $W^I(1,0) - W^{RC} > 0$ if $\alpha > \frac{2D-1}{2(U+D)}$; and this holds, because $\frac{2D-1}{2(U+D)} < \overline{\alpha}$. An analogous argument holds if $\alpha < \underline{\alpha}$. Suppose $\alpha \in (\underline{\alpha}, \overline{\alpha})$. For simplicity assume $\beta = \frac{1}{4}$ (a similar argument holds if $\beta \in (0, \frac{1}{4})$). Then

$$W^I(\hat{p}, \hat{q}) - W^{RC} = \alpha\left(\hat{p} - \frac{1}{2}\right)(\pi_{Aa} - \pi_{Ab}) + (1-\alpha)\left(\hat{q} - \frac{1}{2}\right)(\pi_{Bb} - \pi_{Ba}) = (2D-1)(2U-1) > 0.$$

■

4 Biased sampling

In this section we allow for *homophily* (bias towards sampling same type individuals) and *heterophily* (bias towards sampling other type individuals). Homophilic tendencies are widely documented in the social networks literature (see Currarini, Jackson and Pin 2009 and references therein), and may be due to segregation or individuals' preferences for having friends that are similar to them.

We introduce homophily and heterophily in our model using the parameters $\alpha_A \in (0,1)$ and $\alpha_B \in (0,1)$, which correspond to the the relative intensities with which type A individuals sample a type A individual and type B individuals sample a type B individual, respectively. In the benchmark case analyzed above, $\alpha_A = \alpha$ and $\alpha_B = 1-\alpha$; here, if $\alpha_A > (<)\alpha$, then type A individuals are homophilous (heterophilous); and similarly, if $\alpha_B > (<)1-\alpha$ then type B individuals are homophilous (heterophilous).

General intensities. In order to accommodate biased sampling we need to generalize the benchmark model allowing the sampling intensities to differ from the fraction of the population in each state. We refer to this case as the model with *general intensities.* In particular, we define the function $\theta_{\sigma\sigma'} : \Delta(\Sigma) \to [0,1]$, where $\theta_{\sigma\sigma'}((p_{\sigma''})_{\sigma''\in\Sigma})$ is the intensity with which a state σ individual samples a state σ' individual when the cross-sectional distribution of individual states is $(p_{\sigma''})_{\sigma''\in\Sigma}$, for all $\sigma, \sigma' \in \Sigma$.

Mass-balancing condition. Since our analysis requires that individuals sample each other, our model needs to satisfy the *mass-balancing condition*

$$p_\sigma \theta_{\sigma\sigma'}((p_{\sigma''})_{\sigma''\in\Sigma}) = p_{\sigma'}\theta_{\sigma'\sigma}((p_{\sigma''})_{\sigma''\in\Sigma}),$$

for all $(p_{\sigma''})_{\sigma''\in\Sigma} \in \mathit{\Delta}(\Sigma)$ and $\sigma, \sigma' \in \Sigma$. The mass-balancing condition is satisfied automatically in the benchmark model analyzed above, because there we assume that intensities are equal to the fraction of individuals in that individual state within the population; i.e., $\theta_{\sigma\sigma'}((p_{\sigma''})_{\sigma''\in\Sigma}) = p_{\sigma'}$, for all $(p_{\sigma''})_{\sigma''\in\Sigma} \in \mathit{\Delta}(\Sigma)$ and $\sigma, \sigma' \in \Sigma$.[23] However, the mass-balancing condition does not hold in general in the model with general intensities. The construction provided here tackles this problem by introducing an additional variable in the individual state. This binary variable represents a dummy-sampling type, which allows some individuals to sample without changing their actions (that is, these individuals are only observed by other individuals). Thus, by introducing dummy-sampling types, we are able to satisfy the mass-balancing condition while at the same time effectively generate biased sampling.

Individual states. We extend the definition of *individual states* including the binary variable $\epsilon \in \{0, 1\}$; thus, we now consider individual states $\sigma := (\tau, c, x, \delta_A, \delta_B, \epsilon) \in T \times S \times X \times \Delta^2 \times \{0, 1\}$. Sampling affects an individual, in the sense that she will update her choice according to the decision rule L (provided that the observed action is different from hers), only if her state features $\epsilon = 1$.

Individual state changes. If the individual state σ is such that $\epsilon = 0$, then type τ and action c remain the same upon sampling and the new state is $(\tau, c, x'', \delta''_A, \delta''_B, \epsilon'')$ with probability $\mu_{\tau c}(x'')\mu_{\tau Ad}(\delta''_A)\mu_{\tau Bd}(\delta''_B)\mu_{\epsilon_\tau}(\epsilon'')$, where $\mu_{\epsilon_\tau}(\epsilon'')$ is a constant in $[0, 1]$ for $\tau \in \{A, B\}$ and $\epsilon'' \in \{0, 1\}$. Otherwise, if $\epsilon = 1$, an individual whose state is $\sigma = (\tau, c, x, \delta_A, \delta_B, \epsilon)$ and has sampled an individual with state $\sigma' = (\tau', c', x', \delta'_A, \delta'_B, \epsilon')$ updates her state to $\sigma'' = (\tau'', c'', x'', \delta''_A, \delta''_B, \epsilon'')$ with probability $\varsigma_{\sigma\sigma'}(\sigma'')$, given by

$$\varsigma_{\sigma\sigma'}(\sigma'') = \begin{cases} \mu_{\tau c}(x'')\mu_{\tau Ad}(\delta''_A)\mu_{\tau Bd}(\delta''_B)\mu_{\epsilon_\tau}(\epsilon'') & \text{if} \quad c = c' = c'' \text{ and } \tau'' = \tau \\ L(x, x', \delta_{\tau'})\mu_{\tau c'}(x'')\mu_{\tau Ac}(\delta''_A)\mu_{\tau Bc}(\delta''_B)\mu_{\epsilon_\tau}(\epsilon'') & \text{if} \quad c'' = c' \neq c \text{ and } \tau'' = \tau \\ (1 - L(x, x', \delta_{\tau'}))\, \mu_{\tau c}(x'')\mu_{\tau Ac'}(\delta''_A)\mu_{\tau Bc'}(\delta''_B)\mu_{\epsilon_\tau}(\epsilon'') & \text{if} \quad c'' = c \neq c' \text{ and } \tau'' = \tau \\ 0 & \text{otherwise,} \end{cases}$$

where $d \neq c$. In order to have mass-balance, if $\alpha(1 - \alpha_A) < (1 - \alpha)(1 - \alpha_B)$, then type A individuals are required to have dummy-samples, whereas if $\alpha(1 - \alpha_A) > (1 - \alpha)(1 - \alpha_B)$, then type B individuals are required

[23] And this condition does play a role in the analysis, because the proof of Proposition 4 in Appendix B uses Corollaries 1 and 2 in Duffie, Qiao and Sun (2016), which assume that the mass-balancing condition holds.

to have dummy-samples. Hence, we set

$$
\begin{aligned}
\mu_{\epsilon_A}(1) &= \min\left\{1, \frac{\alpha(1-\alpha_A)}{(1-\alpha)(1-\alpha_B)}\right\} = 1-\mu_{\epsilon_A}(0)\\
\mu_{\epsilon_B}(1) &= \min\left\{1, \frac{(1-\alpha)(1-\alpha_B)}{\alpha(1-\alpha_A)}\right\} = 1-\mu_{\epsilon_B}(0).
\end{aligned}
$$

Intensities. We set the intensities to

$$
\theta_{\sigma\sigma'}((p_{\sigma''})_{\sigma''\in\Sigma}) = \begin{cases}
\frac{1}{\mu_{\epsilon_A}(1)}\alpha_A\alpha^{-1}p_{\sigma'} & \text{if} \quad \tau=\tau'=A\\
\frac{1}{\mu_{\epsilon_A}(1)}(1-\alpha_A)(1-\alpha)^{-1}p_{\sigma'} & \text{if} \quad \tau=A \text{ and } \tau'=B\\
\frac{1}{\mu_{\epsilon_B}(1)}(1-\alpha_B)\alpha^{-1}p_{\sigma'} & \text{if} \quad \tau=B \text{ and } \tau'=A\\
\frac{1}{\mu_{\epsilon_B}(1)}\alpha_B(1-\alpha)^{-1}p_{\sigma'} & \text{if} \quad \tau=\tau'=B,
\end{cases}
$$

for all $(p_{\sigma''})_{\sigma''\in\Sigma} \in \Delta(\Sigma)$. Thus, the model satisfies the mass-balancing condition.

The differential equations, analogous to (4)-(5), that drive the dynamical system in presence of homophily and/or heterophily are[24]

$$
\begin{aligned}
\alpha\dot{p} &= \alpha(1-p)\left[\alpha_A p + (1-\alpha_A)(1-q)\right]L_{ba}(A) - \alpha p\left[\alpha_A(1-p)+(1-\alpha_A)q\right]L_{ab}(A) && (10)\\
(1-\alpha)\dot{q} &= (1-\alpha)(1-q)\left[\alpha_B q + (1-\alpha_B)(1-p)\right]L_{ab}(B) - (1-\alpha)q\left[\alpha_B(1-q)+(1-\alpha_B)p\right]L_{ba}(B) && (11)
\end{aligned}
$$

In Subsection 10.1 of Appendix D we explain how to derive these differential equations. We now provide the analysis of the dynamics implied by this system.

Stable Equilibria. The resulting dynamics are qualitatively similar to the case of uniform sampling, yet the analysis allows us to obtain insights about the effect of these biases on the population's choices. As before, we obtain convergence to either a corner rest point or a unique interior rest point. We also obtain some results, however, that cannot arise under uniform sampling. The corner rest point that is non-optimal for a type of individuals can be ruled out if this type is sufficiently homophilous. More generally, the fraction of both types of individuals choosing a (corresp. b) increases in α_A (corresp. α_B). This means that a type benefits from being more homophilous and is affected negatively by the homophily of the other type. In the limit, as α_A and α_B go to one, so each type is completely homophilous, the global attractor of the system approaches $(1,1)$, i.e., the entire population makes the right choice. Hence, the limit of the model when α_A and α_B go to one corresponds to the case of two homogeneous populations.

[24]In formal terms, Proposition 4 of Appendix B holds in the model with general intensities as well (with the natural necessary adjustment to the definition of $\hat{\Delta}(\Sigma)$ to accommodate for the dummy-sampling variable in the individual state).

Formally, fix $U, D \in (\frac{1}{2}, \frac{3}{4}]$, and define the functions $\overline{\alpha}_A : \mathbb{R} \to \mathbb{R}$ and $\overline{\alpha}_B : \mathbb{R} \to \mathbb{R}$ such that

$$\overline{\alpha}_A(z) = \frac{D - U + zD(2U-1)}{(1-U)(2D-1)} \text{ and } \overline{\alpha}_B(z) = \frac{U - D + zU(2D-1)}{(1-D)(2U-1)},$$

respectively, for all $z \in \mathbb{R}$. It is easy to see that $\overline{\alpha}_A > \left[\overline{\alpha}_B^{-1}\right]$, and $\alpha_A < (>) \left[\overline{\alpha}_B^{-1}\right](\alpha_B)$ if and only if $\alpha_B > (<) \overline{\alpha}_B(\alpha_A)$, where $\left[\overline{\alpha}_B^{-1}\right]$ is the inverse function of $\overline{\alpha}_B$. The following lemma characterizes virtually all the pairs (α_A, α_B) such that $(1,0)$ is asymptotically stable and such that $(0,1)$ is asymptotically stable.[25] Appendix D contains all proofs of this section.

Lemma 2 *Suppose $\alpha_A \neq \overline{\alpha}_A(\alpha_B)$ and $\alpha_B \neq \overline{\alpha}_B(\alpha_A)$. Then, (i) $(1,0)$ is asymptotically stable if and only if $\alpha_A > \overline{\alpha}_A(\alpha_B)$, (ii) $(0,1)$ is asymptotically stable if and only if $\alpha_A < \left[\overline{\alpha}_B^{-1}\right](\alpha_B)$, and (iii) if $(1,0)$ is asymptotically stable, then $(0,1)$ is not asymptotically stable (and vice versa).*

Corollary 1 *(i) If $\alpha_B > \frac{1-D}{D}$, then $(1,0)$ is not asymptotically stable, and if $U > D$ and $\alpha_B < \frac{U-D}{U(2U-1)}$, then $(1,0)$ is asymptotically stable. (ii) If $\alpha_A > \frac{1-U}{U}$, then $(0,1)$ is not asymptotically stable, and if $D > U$ and $\alpha_A < \frac{D-U}{U(2D-1)}$, then $(0,1)$ is asymptotically stable.*

Lemma 2 implies that for any $U, D \in (\frac{1}{2}, \frac{3}{4}]$ and $\alpha_A, \alpha_B \in (0,1)$ either (i) $(1,0)$ is asymptotically stable, (ii) $(0,1)$ is asymptotically stable, or (iii) neither $(1,0)$ nor $(0,1)$ is asymptotically stable. For large values of α_A relative to α_B, $(1,0)$ is asymptotically stable, whereas for large values of α_B relative to α_A, $(0,1)$ is asymptotically stable. For more similar values of α_A and α_B, neither $(1,0)$, nor $(0,1)$ is asymptotically stable. This is illustrated in Figure 2 for $U = 0.7$ and $D = 0.65$. The northwest (corresp. southeast) region corresponds to the parameter values such that $(1,0)$ (corresp. $(0,1)$) is asymptotically stable. In the center region neither $(1,0)$, nor $(0,1)$ is asymptotically stable. Our next result reveals that there is a unique interior rest point if and only if (α_A, α_B) is in the center region of Figure 2.

[25] The only possibilities we do not consider here are when $\alpha_A = \overline{\alpha}_A(\alpha_B)$ or $\alpha_B = \overline{\alpha}_B(\alpha_A)$, in which case some rest points may not be hyperbolic and hence may not be determined using the Jacobian of the system.

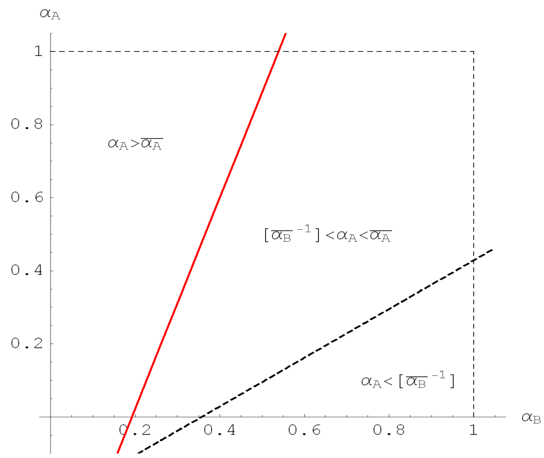

Figure 2. $U = 0.7$ and $D = 0.65$. The solid line is $\overline{\alpha}_A$ and the dashed line $\left[\overline{\alpha}_B^{-1}\right]$.

Lemma 3 *(i) If $\alpha_A < \left[\overline{\alpha}_B^{-1}\right](\alpha_B)$ or $\alpha_A > \overline{\alpha}_A(\alpha_B)$, then there is no interior rest point. (ii) If $\left[\overline{\alpha}_B^{-1}\right](\alpha_B) < \alpha_A < \overline{\alpha}_A(\alpha_B)$, then there is a unique interior rest point.*

Lemmas 2 and 3 establish that, setting aside the cases where rest points may not be hyperbolic, i.e., $\alpha_A = \overline{\alpha}_A(\alpha_B)$ and $\alpha_B = \overline{\alpha}_B(\alpha_A)$, there is an interior rest point if and only if neither $(1,0)$ nor $(0,1)$ is asymptotically stable. The closed-form expression describing the interior rest point, denoted by $(\widetilde{p},\widetilde{q})$, is cumbersome and it is provided in Lemma 8 of Appendix D. The following result provides the global attractors of the system for virtually all the possible values of α_A and α_B.

Proposition 2 *If $(p(0),q(0)) \notin RP$, then*

$$\lim_{t\to\infty}(p(t),q(t)) = \begin{cases} (0,1) & \text{if} \quad \alpha_A < \left[\overline{\alpha}_B^{-1}\right](\alpha_B) \\ (\widetilde{p},\widetilde{q}) & \text{if} \quad \left[\overline{\alpha}_B^{-1}\right](\alpha_B) < \alpha_A < \overline{\alpha}_A(\alpha_B) \\ (1,0) & \text{if} \quad \overline{\alpha}_A(\alpha_B) < \alpha_A. \end{cases}$$

We shall emphasize two implications of the results above. First, together with Corollary 1, Proposition 2 implies that if a given type is sufficiently homophilous, then the population will not converge to choose that type's non-optimal action. For example, if $\alpha_B > \frac{1-D}{D}$ then the system will not converge to $(1,0)$ regardless of α_A and U. An implication is that if both types are sufficiently homophilous, then the population converges to an interior rest point. Second, if $U > D$ and additionally type B is relatively small or heterophilous, so that α_B is below a threshold value (determined by U and D), then the whole population converges to action a, regardless of α_A. These observations are illustrated in Figure 2, where $U > D$. Here if $\alpha_B > 0.54$, then there is no α_A such that the population converges to choose a. On the other hand if $\alpha_B < 0.19$, then the population converges to action a even if α_A is arbitrarily small.[26]

[26]An implication of this result is that uniform adoption does not require uniform sampling. Furthermore, both types being

Comparative Statics. The fraction of a type that chooses its optimal action in an interior rest point increases in that type's homophily. Fix $U, D \in \left(\frac{1}{2}, \frac{3}{4}\right]$ and let $(\widetilde{p}(\alpha_A, \alpha_B), \widetilde{q}(\alpha_A, \alpha_B))$ be the interior rest point, when it exists, as a function of α_A and α_B. We obtain that the partial derivatives of these functions satisfy $\widetilde{p}_1(\alpha_A, \alpha_B) > 0$, $\widetilde{p}_2(\alpha_A, \alpha_B) < 0$, $\widetilde{q}_1(\alpha_A, \alpha_B) < 0$ and $\widetilde{q}_2(\alpha_A, \alpha_B) > 0$ (see Appendix D). Intuitively, interior rest points are determined by the balance of flows into and out of each action for each type. When sampling is uniform, interior rest points (p^*, q^*) satisfy $p^* > 1 - q^*$. Hence in equilibrium, a greater probability of sampling individuals of the same type makes it more likely for an individual to sample her optimal action. This allows a larger fraction of this type of individuals to choose their optimal action, as they are less exposed to the possibility of sampling an individual of the other type choosing the other action.

Finally, from the results above, the system cannot converge to $(1, 0)$ when $\alpha_B > \frac{1-D}{D}$. The following remark, however, reveals that $\widetilde{p}$ can be arbitrarily close to 1, even if $\alpha_B > \frac{1-D}{D}$, provided that type A individuals are sufficiently homophilous. This result also implies that in the limit, as each type becomes completely homophilous, a heterogeneous population behaves as two homogenous populations.

Remark 3 *If* $\alpha_B > \frac{1-D}{D}$, *then* $\lim_{\alpha_A \to 1} \widetilde{p}(\alpha_A, \alpha_B) = 1$ *and if* $\alpha_A > \frac{1-U}{U}$, *then* $\lim_{\alpha_B \to 1} \widetilde{q}(\alpha_A, \alpha_B) = 1.$

5 A monopolist with heterogeneous consumers

In this section we consider a monopolist who serves a market of boundedly rational consumers with heterogeneous valuations for his product. We assume that the expected valuation of the product is positive for some consumers and zero for others. The two possible choices for consumers are buying or not buying the product. The monopolist knows the expected value of the valuation and the size of each type. Furthermore, he knows that consumers are boundedly rational and their decision rules. The valuation of each individual consumer, however, is unknown to both the monopolist and the consumer.

Consumers' decisions. To fix ideas, suppose the population consists of individuals that suffer from a chronic illness that produces unpleasant symptoms. The monopolist provides the only available treatment. Symptoms present themselves randomly and translate into *physical payoffs*, measured in monetary units, which take values within a finite set whose minimum is 0.5 and maximum is 1.[27] The outcomes 0.5 and 1 can be thought of as the physical payoff of "full symptoms" and "no symptoms," respectively. The expected physical

heavily biased toward sampling individuals of their own type is not incompatible with uniform adoption. For instance, in Figure 2, we just require $\alpha_B < 0.19$ for uniform adoption of a to occur for any α_A. More broadly, if we measure type B individuals' bias to sample their own type by $\psi_B := \alpha_B(1-\alpha)^{-1}$, it is easy to see that we can have uniform adoption even if ψ_B is very high: if we fix ψ_B, for sufficiently small $(1-\alpha)$ one obtains that $\psi_B(1-\alpha) < 0.54$ and hence, uniform adoption of a (for high enough α_A).

[27] We normalize the physical payoffs to be in $[0.5, 1]$ so that the total payoff of each choice, corresponding to the physical payoffs minus the price charged by the monopolist, fall in $[0, 1]$, as in the benchmark model.

payoff of an untreated individual is $\gamma < 1$. The treatment of the monopolist is effective for type A individuals but has no effect on type B individuals. The expected physical payoff of a type A individual under treatment is $\gamma' \in (\gamma, 1]$. Therefore, $\varphi := \gamma' - \gamma > 0$ measures the expected effectiveness of the treatment for type A individuals. The expected physical payoff of a type B individual under treatment is γ.[28] Individuals choose whether to buy the treatment from the monopolist (action a) or buy no treatment (action b). The monopolist sells the treatment at price $r \in [0, \varphi]$ to both type A and type B individuals.[29] Each individual's payoff is equal to her physical payoff minus the price paid to the monopolist. Hence, individuals' payoff fall in $[0, 1]$. Expected payoffs are, thus, given by $\pi_{Aa} = \gamma' - r$, $\pi_{Ab} = \gamma$, $\pi_{Ba} = \gamma - r$ and $\pi_{Bb} = \gamma$.

As before, each individual makes choices after observing her own payoff, that of another individual, and a comparison signal. We assume that the price the monopolist charges is observed and individuals know that they are all charged the same price if they buy the treatment. Therefore, the comparison signal is informative only about the difference in expected physical payoffs under the observed individual's choice.[30] Formally, the expected values of the comparison signals are $\pi_{ABa} = \gamma' - \gamma$, $\pi_{BAa} = \gamma - \gamma'$, $\pi_{AAa} = 0$, $\pi_{BBa} = 0$, and $\pi_{\tau\tau' b} = 0$ for $\tau, \tau' \in \{A, B\}$. In order to make the interpretation of the comparison-signal consistent with previous sections, and since physical payoffs are contained in $[0.5, 1]$, we restrict the finite set of comparison-signals to have minimum -0.5 and maximum 0.5. The individuals' decision rule is $L(x, y, \delta) = \frac{1}{2} + \beta(y + \delta - x)$ as before. Since here $\min_{(x,y,\delta)\in X^2\times\Delta} \{y + \delta - x\} \geq -1.5$ and $\max_{(x,y,\delta)\in X^2\times\Delta} \{y + \delta - x\} \leq 1.5$, we can now choose β from $(0, 1/3]$.[31] For simplicity we assume $\beta = 1/3$ and obtain $L_{ba}(A) = \frac{1}{2} + \frac{1}{3}(\varphi - r) =: \widehat{U}(\varphi, r)$ and $L_{ab}(B) = \frac{1}{2} + \frac{1}{3}r =: \widehat{D}(\varphi, r)$. Hence, $L_{ba}(A), L_{ab}(B) \in \left[\frac{1}{2}, \frac{2}{3}\right]$.

The demand curve. We assume that the monopolist fully understands how individuals make choices, although he does not know the type of each individual or the realizations of payoffs and comparison signals. We also assume that the monopolist chooses a fixed price that maximizes his profit in the asymptotically stable

[28]Our model is related to Spiegler's (2006) model of markets for "quacks." In his paper the treatment is ineffective for all individuals in the population, so he refers to the healer as a quack. In our model the healer is not a quack since the treatment works for some individuals, although it is ineffective for others.

[29]At prices above φ, under full information, both type A and type B individuals prefer not to buy the treatment. In our setting, such prices would lead the population to a state in which no individual buys the treatment. As we show below, prices larger than φ do not add anything to the analysis.

[30]For example, suppose the illness is chronic overweight and the treatment is an individualized diet prescribed by a nutritionist. Whether the diet is effective for weight control may depend, among other things, on individuals' willpower. Such considerations may lead an individual to see the nutritionist even after hearing about the negative experience of an undisciplined acquaintance, if she believes herself to have more willpower than the acquaintance.

[31]From footnote 12, we need $\beta \in \left(0, \frac{1}{2\overline{c}}\right]$ and here $y + \delta - x \in [-\overline{c}, \overline{c}] = [-1.5, 1.5]$.

rest point of the dynamic system (p^*, q^*).[32] The monopolist, thus, faces a demand curve given by

$$Q(r) := \alpha p^* + (1-\alpha)(1-q^*),$$

where (p^*, q^*), as explained below, is determined by r, φ, and α.

We can find the prices such that $(p^*, q^*) = (1, 0)$ and $(p^*, q^*) = (0, 1)$, respectively. In order to do this, we need to find the values of r such that $\underline{\alpha}(\widehat{U}(\varphi, r), \widehat{D}(\varphi, r)) = \alpha$ and $\overline{\alpha}(\widehat{U}(\varphi, r), \widehat{D}(\varphi, r)) = \alpha$, respectively. We denote these values by $\overline{r}(\varphi, \alpha)$ and $\underline{r}(\varphi, \alpha)$, respectively. Simple computations reveal that

$$\begin{aligned} \underline{r}(\varphi, \alpha) &= \frac{1}{4}(3 + 2\varphi - \sqrt{(3+2\varphi)^2 - 24\alpha\varphi}) \\ \overline{r}(\varphi, \alpha) &= \frac{1}{4}(2\varphi - 3 + \sqrt{(3-2\varphi)^2 + 24\alpha\varphi}). \end{aligned}$$

Note that $0 < \underline{r}(\varphi, \alpha) < \overline{r}(\varphi, \alpha) < \varphi$. Further, since $\widehat{U}(\varphi, r)$ and $\widehat{D}(\varphi, r)$ are strictly decreasing and strictly increasing in r, respectively, and $\overline{\alpha}$ is strictly decreasing in its first argument and strictly increasing in its second argument, it follows that the composition $\overline{\alpha}(\widehat{U}(\varphi, r), \widehat{D}(\varphi, r))$ is a strictly increasing function of r. Therefore, if $r \leq \underline{r}(\varphi, \alpha)$, then $\overline{\alpha}(\widehat{U}(\varphi, r), \widehat{D}(\varphi, r)) \leq \alpha$, and by Theorem 1 $(p^*, q^*) = (1, 0)$. By an analogous argument, if $r \geq \overline{r}(\varphi, \alpha)$, then $(p^*, q^*) = (0, 1)$.[33] Otherwise, if the monopolist sets a price level $r \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$, then the asymptotically stable rest point will be interior and it will be given by $(p^*, q^*) = \Big(\hat{p}(\alpha, \widehat{U}(\varphi, r), \widehat{D}(\varphi, r)), \hat{q}(\alpha, \widehat{U}(\varphi, r), \widehat{D}(\varphi, r))\Big)$, where $\hat{p}$ and $\hat{q}$ are the functions defined in Section 3. Thus, the demand curve faced by the monopolist, $Q : [0, \infty) \to [0, 1]$, is given by

$$Q(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \underline{r}(\varphi, \alpha) \\ \frac{2r^2 + r(3-2\varphi) - 3\alpha\varphi}{4r(r-\varphi)} & \text{if } \underline{r}(\varphi, \alpha) < r < \overline{r}(\varphi, \alpha) \\ 0 & \text{if } r \geq \overline{r}(\varphi, \alpha). \end{cases}$$

If the price is below a certain threshold the monopolist sells to the entire population. If it is above a certain threshold no one buys the treatment in the long run. At intermediate prices the monopolist captures a fraction of the population of both types. In this case, the demand curve is downward sloping and converges to 0 and 1 as the price goes to $\overline{r}(\varphi, \alpha)$ and $\underline{r}(\varphi, \alpha)$, respectively.

[32] This simplifies the analysis significantly. We leave the alternative, in which the monopolist has a positive discount rate and maximizes the present value of all his future profits, for future research.

[33] The statement of Theorem 1 does not include the cutoff values, which in this case correspond to $r = \underline{r}(\varphi, \alpha)$ and $r = \overline{r}(\varphi, \alpha)$, since the techniques used to prove Theorem 1 are valid for hyperbolic rest-points and some rest points are non-hyperbolic at the cutoff values. Nevertheless, by directly applying the definition of asymptotic stability and constructing a straightforward $\varepsilon - \gamma$ argument, it can be shown that for the prices $r = \underline{r}(\varphi, \alpha)$ and $r = \overline{r}(\varphi, \alpha)$, (p^*, q^*) is equal to $(1, 0)$ and $(0, 1)$, respectively.

The monopolist optimal prices. We assume that the marginal costs of the monopolist are constant and equal to $k \geq 0$. The profit of the monopolist as a function of the price, $G : [0, \infty) \to \mathbb{R}$, is, hence, given by

$$G(r) = \begin{cases} r - k & \text{if } 0 \leq r \leq \underline{r}(\varphi, \alpha) \\ (r-k)\frac{2r^2 + r(3-2\varphi) - 3\alpha\varphi}{4r(r-\varphi)} & \text{if } \underline{r}(\varphi, \alpha) < r < \overline{r}(\varphi, \alpha) \\ 0 & \text{if } r \geq \overline{r}(\varphi, \alpha). \end{cases}$$

The elasticity of the demand curve drops to zero at $r = \underline{r}(\varphi, \alpha)$, price at which the monopolist would sell the treatment to the whole market. Furthermore, when marginal costs are low enough, the right derivative of the profit function with respect to the price is negative at $r = \underline{r}(\varphi, \alpha)$ and, thus, as we show below, the monopolist chooses $r = \underline{r}(\varphi, \alpha)$. On the other hand, when the marginal costs are higher, he charges a higher price and sells the treatment to only a fraction of the population. More precisely, we have the following relation between marginal costs and the change in profits at the price at which the monopolist sells to the whole population. Let $G'(x^+)$ denote the right derivative of G for all $x \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$.

Lemma 4 *There is a unique marginal cost $\hat{k} \in (0, \underline{r}(\varphi, \alpha))$ such that $G'(\underline{r}(\varphi, \alpha)^+) = 0$. Further, $G'(\underline{r}(\varphi, \alpha)^+) < 0$ if $k < \hat{k}$ and $G'(\underline{r}(\varphi, \alpha)^+) > 0$ if $k > \hat{k}$.*

Proof. If $k = 0$, then

$$G'(\underline{r}(\varphi, \alpha)^+) = 1 + \underline{r}(\varphi, \alpha) Q'(\underline{r}(\varphi, \alpha)^+) = \frac{2\varphi - 3}{2\varphi - 3 + \sqrt{(3 + 2\varphi)^2 - 24\alpha\varphi}} < 0.$$

If $k = \underline{r}(\varphi, \alpha)$, then $G'(\underline{r}(\varphi, \alpha)^+) = 1 > 0$. Simple computations reveal that $G'(r)$ is continuous and increasing in k for all $r \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$. Hence, there is a unique $\hat{k} \in (0, \underline{r}(\varphi, \alpha))$ such that $G'(\underline{r}(\varphi, \alpha)^+) = 1 + (\overline{r}(\varphi, \alpha) - \hat{k}) Q'(\underline{r}(\varphi, \alpha)^+) = 0$. Further, by the monotonicity of $G'(\underline{r}(\varphi, \alpha)^+)$ in k, $G'(\underline{r}(\varphi, \alpha)^+) < 0$ if $k < \hat{k}$ and $G'(\underline{r}(\varphi, \alpha)^+) > 0$ if $k > \hat{k}$. ■

The next result characterizes the optimal prices charged by the monopolist. The proof is provided in Appendix E.

Proposition 3 *Let r^* be the price that maximizes the monopolist's profits. If $k \leq \hat{k}$, then $r^* = \underline{r}(\varphi, \alpha)$. If $k \in (\hat{k}, \overline{r}(\varphi, \alpha))$, then r^* is the unique solution to $G'(r) = 0$ and it is contained in $(\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$. Otherwise, if $k \geq \overline{r}(\varphi, \alpha)$, then the monopolist withdraws from the market.*

If the marginal cost is low enough, when $k < \hat{k}$, the monopolist sets the highest price at which he sells to the whole population. At a higher price he loses both type A and type B clients. In this case, the reduction in sales outweighs the positive effect on profits of the higher price. If the marginal cost is sufficiently high, when $k > \overline{r}(\varphi, \alpha)$, the monopolist chooses not to produce. In this case, marginal costs may well be below type A individuals' willingness to pay, which occurs if $k \in (\overline{r}(\varphi, \alpha), \varphi)$. Here, at any price at which the monopolist earns positive profits by selling the treatment, it is unattractive for type B individuals and not sufficiently attractive for type A individuals to the point that buying the treatment does not survive in the long run. Finally, when marginal costs are at an intermediate level, the monopolist sells his treatment to a positive fraction of both type A and type B individuals.

Figure 3 shows the inverse demand and marginal revenue curves of the monopolist (as a function of the quantity). If the marginal cost is low, it is not intersected by the marginal revenue curve. The monopolist then chooses a corner solution and charges the highest price at which he captures the entire market (the left panel of Figure 3). When marginal costs are in the intermediate range the quantity sold in the market equates marginal cost with marginal revenue (the right panel of Figure 3). If the marginal cost is sufficiently high, the monopolist withdraws from the market.

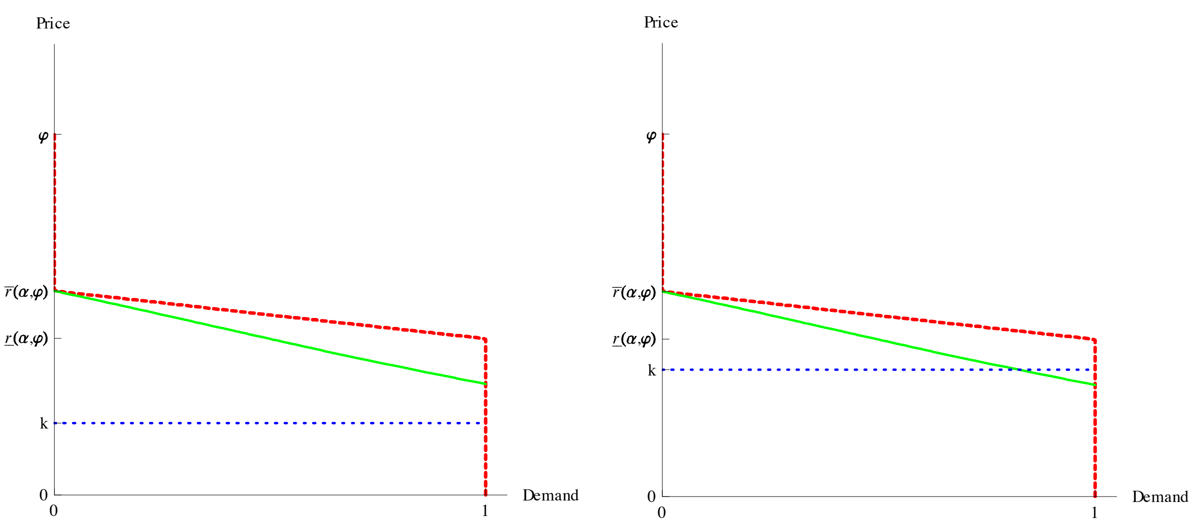

Figure 3: The dashed line is the inverse demand curve, the solid line is marginal revenue and the dotted line is marginal cost. $\alpha = 1/2$, $\varphi = 0.4$, $k = 0.08$ (left) and $k = 0.14$ (right).

Whenever $\hat{k} < \overline{r}(\varphi, \alpha)$ the optimal price is strictly increasing in both α and φ. With a larger α the dynamics towards buying the treatment are stronger due to the sampling effects discussed before. With a larger φ type A individuals tend to choose treatment more often while the sensitivity of type B individuals' problem is unaffected. In both of these cases the monopolist loses fewer clients when raising the price and therefore charges a higher price in equilibrium. The formal arguments for these comparative statics appear in Appendix E.

Welfare analysis. Under full information, i.e., if the monopolist and individuals knew their type, the monopolist would set the price at φ and only type A individuals would buy the treatment. There would hence be no deadweight loss in a standard computation of consumer and producer surplus. Further, the monopolist would earn the entire surplus, equal to $\alpha(\varphi - k)$. When individuals are boundedly rational and the monopolist sets the price at r^*, (i) there is always a deadweight loss (except in the cases $k = 0$ and $k \geq \varphi$), (ii) the monopolist earns lower profits than under full information, and (iii) type A and type B individuals are weakly better off and weakly worse off, respectively, than under full information. The deadweight loss in our setup comes from two different sources. First, whenever the treatment is sold in the market (i.e., whenever $k < \overline{r}(\varphi, \alpha)$) the cost of providing type B individuals with treatment causes a deadweight loss. Second, whenever $k \in (\hat{k}, \varphi)$ a deadweight loss arises since some type A individuals do not buy treatment in spite of having a willingness to pay that exceeds the marginal cost.

The profits of the monopolist are smaller in our setup compared to the case of full information.[34] Intuitively, type B individuals tend to not buy the treatment in their individual choices and this influences type A individuals through sampling effects. Therefore, in comparison to the case of full information, the monopolist must lower his price to sell the treatment. Even if he sells the treatment to the entire population, the negative effect of the reduced price outweighs the positive effect of increased sales.

Type A individuals are weakly better off in our setup since the monopolist extracts the entire surplus in the case of full information. In contrast, here he may set a lower price, allowing type A individuals to obtain a positive consumer surplus. Type B individuals are weakly worse off in our setup, since they may end up buying a worthless treatment.

Biased sampling. While for simplicity the results here are obtained under the assumption of uniform sampling, the analysis extends in a relatively straightforward manner to biased sampling (at the cost of slightly more involved derivations). Under biased sampling $\overline{\alpha}_A(z)$ and $\overline{\alpha}_B(z)$ define cutoff prices in an analogous way as above. The homophily of each type affects the cutoff prices in the obvious way. Demand depends positively on the homophily of type A and negatively on the homophily of type B and the monopolist's profit follows the same pattern. In the limit, when both types are perfectly homophilic, the monopolist sets some price very close to φ and sells only to the entire type A population. In other words, perfect homophily brings us back to the case of full information.

[34]For the formal argument, see Remark 8 in Appendix E.

6 Discussion

In the benchmark model, we assume that individuals' assessments of their expected payoff relative to others are unbiased. Individuals often make systematic mistakes in assessments of relative abilities.[35] In Appendix F, we illustrate the consequences of these mistakes by introducing biases in the comparison signal. The results we obtain are qualitatively similar to those of the benchmark model. An analysis of this extension, however, allows us to assess the effect of biased comparison signals on the long-run equilibrium. We show that if type τ individuals have *positive biases*, i.e., $\pi_{\tau\tau' c} > \pi_{\tau c} - \pi_{\tau' c}$ for all $\tau' \in T$ and $c \in S$, this leads to a smaller fraction of that type choosing their optimal action in the long run. We also consider the possibility of *negative biases* and show that this leads to better long-run outcomes. Intuitively, if individuals have positive biases, they switch more often to both their optimal and suboptimal action. Since in equilibrium most individuals of at least one of the types are choosing their optimal action, more switching leads to excessive switching away from the optimal action. In contrast, negative biases cause individuals to be more reluctant to switch and, hence, to choose their optimal action more often. The details are provided in Appendix F.

Since we consider linear decision rules, there is no role for the accuracy of the comparison signal in our setup. It is intuitive, however, that the experiences of people who are different are less informative. Some empirical evidence suggests that information about different individuals is often discarded (see, e.g., Munshi 2004). This could be handled by assuming that the variance of the comparison signal is greater when observing a different type and by considering a concave decision rule. The experiences of different individuals would then be discounted relative to those of similar individuals. Therefore, there would be less switching towards the actions chosen by individuals of the other type. This could lead to results similar to those of biased comparisons. We leave a thorough analysis for future research.

Our model assumes an exogenous sampling process. In the presence of heterogeneity, however, individuals may have incentives to search for individuals that are similar to them and hence more suitable to learn from. At the same time, it seems that individuals would prefer to sample others who have made good choices (e.g., Offerman and Schotter 2009). An analysis that considers an endogenous sampling process might provide interesting insights into the implications of heterogeneity in the search for suitable role models.

Our model is qualitatively consistent with several empirical aspects of the process of diffusion of innovations. In particular, it yields S-shaped adoption curves as those found in the empirical evidence. Furthermore, our model is consistent with some features of the diffusion of hybrid corn in Kenya found by Suri (2011): less-than-full final adoption, heterogeneities in returns to adoption, and equilibrium switching behavior. A quantitative

[35] For example, individuals tend to overestimate their abilities and performance at different tasks, a phenomenon that is known as positive self-image or overconfidence (see, e.g., Van den Steen 2004, Santos-Pinto and Sobel 2005).

analysis is beyond the scope of this paper and is left for future research.

7 Appendix A: Proof of Necessity in Proposition 1

We define the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ with $\hat{\Omega} = X^2 \times \Delta$, $\hat{\mathcal{F}} = 2^{X^2\times\Delta}$, and the probability measure induced by $\mu_{\tau c,\tau' d}$, which we call the *environment*. The (marginal) distribution functions $\mu_{\tau c}$, $\mu_{\tau' d}$ and $\mu_{\tau\tau' d}$ are derived from $\mu_{\tau c,\tau' d}$ in the usual way. Thus, the payoff distribution of a type τ individual who chooses c is $\mu_{\tau c}$ and the probability of the event $\{x \in D\}$, with $D \subseteq X$, is denoted by $\mu_{\tau c}(D)$; and if D is a singleton $\{x\}$, with $x \in X$, then its probability is denoted by $\mu_{\tau c}(x)$. The analogous notation conventions apply to the distributions of the comparison signal and the payoff distribution of a type τ' individual who chooses d. Necessity in Proposition 1 is argued using the following lemmata.

Lemma 5 *Suppose L is payoff-ordering. Then, $\pi_{\tau d} = \pi_{\tau c}$ implies $L_{cd}(\tau, \tau') = \frac{1}{2}$ for all different actions c, d and types $\tau, \tau' \in T$.*

Proof. Consider the environment induced by $\mu_{\tau c,\tau' d}$ such that $\pi_{\tau d} = \pi_{\tau c}$ and assume $L_{cd}(\tau, \tau') < \frac{1}{2}$ (the argument for the case $L_{cd}(\tau, \tau') > \frac{1}{2}$ is analogous). We will now consider a different environment induced by a slightly different probability mass, $\widetilde{\mu}_{\tau c,\tau' d}$. Suppose that payoffs and comparison-signals, in both environments, are independent. The modified version of $\mu_{\tau d}$, denoted by $\widetilde{\mu}_{\tau d}$, is such that for any set $I \subset X \setminus \{1\}$ we have $\widetilde{\mu}_{\tau d}(I) = (1-\varepsilon)\mu_{\tau d}(I)$ and $\widetilde{\mu}_{\tau d}(1) = \mu_{\tau d}(1) + \varepsilon\mu_{\tau d}(X \setminus \{1\})$ for some $\varepsilon \in (0, 1]$. Thus, $\widetilde{\pi}_{\tau d} = (1-\varepsilon)\pi_{\tau d} + \varepsilon$. The modified version of $\mu_{\tau c}$, denoted by $\widetilde{\mu}_{\tau c}$, is such that for any $I \subset X \setminus \{0\}$ we have $\widetilde{\mu}_{\tau c}(I) = (1-\varepsilon)\mu_{\tau c}(I)$ and $\widetilde{\mu}_{\tau c}(0) = \mu_{\tau c}(0) + \varepsilon\mu_{\tau c}(X \setminus \{0\})$. Thus, $\widetilde{\pi}_{\tau c} = (1-\varepsilon)\pi_{\tau c}$. The expected value, with the modified distribution, of the comparison signal of a type τ individual who observes a type τ' individual who chooses d, denoted by $\widetilde{\pi}_{\tau\tau' d}$, is

$$
\begin{aligned}
\widetilde{\pi}_{\tau\tau' d} &= (1-\varepsilon)\pi_{\tau d} + \varepsilon - \pi_{\tau' d} \\
&= (1-\varepsilon)\left(\pi_{\tau d} - \pi_{\tau' d}\right) + \varepsilon(1 - \pi_{\tau' d}) \\
&= (1-\varepsilon)\pi_{\tau\tau' d} + \varepsilon(1 - \pi_{\tau' d}),
\end{aligned}
$$

where $\pi_{\tau\tau' d}$ is the expected value of the comparison signal in the initial environment. Suppose that the distribution of this comparison signal in the modified environment is given by a compounded distribution which weights with probabilities $1-\varepsilon$ and ε the distribution of the comparison signal in the initial environment and a degenerate distribution which assigns all the probability to $1 - \pi_{\tau' d}$, respectively. In all the other respects, the modified and initial environments are the same. Let $\widetilde{L}_{cd}(\tau, \tau')$ denote the expected value of the probability of switching to d when $i \in \tau$ chooses c, observes $j \in \tau'$ who chooses d and the comparison signal, in the modified environment. Then, $\widetilde{L}_{cd}(\tau, \tau')$ can be written as a continuous function of ε over the domain $[0, 1]$ and when $\varepsilon = 0$, $\widetilde{L}_{cd}(\tau, \tau') = L_{cd}(\tau, \tau') < \frac{1}{2}$. In order to see this, notice that

$$
\begin{aligned}
\widetilde{L}_{cd}(\tau, \tau') &= \int\int\int L(x, y, \delta) d\widetilde{\mu}_{\tau c}(x) d\widetilde{\mu}_{\tau\tau' d}(\delta) d\widetilde{\mu}_{\tau' d}(y) \\
&= \int\int \left[\varepsilon L(0, y, \delta) + (1-\varepsilon)\int L(x, y, \delta) d\mu_{\tau c}(x)\right] d\widetilde{\mu}_{\tau\tau' d}(\delta) d\mu_{\tau' d}(y) \\
&= \int \left\{(1-\varepsilon)\int\left[\varepsilon L(0, y, \delta) + (1-\varepsilon)\int L(x, y, \delta) d\mu_{\tau c}(x)\right] d\mu_{\tau\tau' d}(\delta) + \right. \\
&\qquad \left. \varepsilon\left[\varepsilon L(0, y, 1-\pi_{\tau' d}) + (1-\varepsilon)\int L(x, y, 1-\pi_{\tau' d}) d\mu_{\tau c}(x)\right]\right\} d\mu_{\tau' d}(y),
\end{aligned}
$$

where the right hand side is a polynomial function of ε. Thus, for small enough ε, $\widetilde{L}_{cd}(\tau, \tau') < \frac{1}{2}$ and, since $\widetilde{\pi}_{\tau d} > \widetilde{\pi}_{\tau c}$, L is not payoff-ordering. ■

Corollary 2 *If L is payoff-ordering, $x, y \in [0, 1]$, $\delta \in [-1, 1]$, and $x = y + \delta$, then $L(x, y, \delta) = \frac{1}{2}$.*

Proof. Consider x, y, δ which satisfy the hypothesis and an environment in which $\mu_{\tau c}(x) = \mu_{\tau d}(y + \delta) = \mu_{\tau' d}(y) = \mu_{\tau\tau' d}(\delta) = 1$.[36] In this environment $L_{cd}(\tau, \tau') = L(x, y, \delta)$, and thus, Lemma 5 implies $L(x, y, \delta) = \frac{1}{2}$. ■

Lemma 6 *If L is payoff-ordering, then $L(x, y, \delta)$ is an affine transformation of $y + \delta - x$ for all $x, y \in [0, 1]$ and $\delta \in [-1, 1]$.*

Proof. Consider arbitrary $x, y \in [0, 1]$ and $\delta \in [-1, 1]$. Consider an environment such that, with probability $\frac{1}{2}$, a first event occurs in which the payoff received by type τ when she chooses c is x, the payoff received by type τ' individual when she chooses d is y, and the comparison signal observed by a type τ individual when she observes the type τ' individual is δ. Otherwise, a second event occurs in which the payoff received by a type τ individual when she chooses c is y, the payoff received by a type τ' individual when she chooses d is x, and the comparison signal observed by a type τ individual when she observes a type τ' individual is $-\delta$. It follows that $\pi_{\tau c} = \pi_{\tau d} = \frac{x+y}{2}$. If L is payoff ordering, then $L_{cd}(\tau, \tau') = \frac{1}{2}$, i.e.,

$$\frac{1}{2}L(x, y, \delta) + \frac{1}{2}L(y, x, -\delta) = \frac{1}{2}. \tag{12}$$

Now consider a second environment that differs from the previous one only in that the first event is replaced by two events that occur with probability $\frac{1}{2}\frac{y+\delta-x+2}{4}$ and $\frac{1}{2}\left(1 - \frac{y+\delta-x+2}{4}\right)$, respectively. In the first of these events,

[36] Notice that since the payoff-ordering property has to hold for arbitrary finite sets $X \subset [0, 1]$ and $\Delta \subset [-1, 1]$, the triplet (x, y, δ) can be chosen arbitrarily within $[0, 1] \times [0, 1] \times [-1, 1]$ (as long as it satisfies the hypothesis of the corollary).

the payoff received by a type τ individual when she chooses c is 0, the payoff received by a type τ' individual when she chooses d is 1, and the comparison signal observed by a type τ individual when she observes a type τ' individual is 1. In the second of these events, the payoff received by a type τ individual when she chooses c is 1, the payoff received by a type τ' individual when she chooses d is 0, and the comparison signal observed by a type τ individual when she observes a type τ' individual is -1. As in the previous environment, $\pi_{\tau c} = \pi_{\tau d}$ and, if L is payoff ordering, then $L_{cd}(\tau, \tau') = \frac{1}{2}$, i.e.,

$$\frac{1}{2}\frac{y+\delta-x+2}{4}L(0,1,1)+\frac{1}{2}\left(1-\frac{y+\delta-x+2}{4}\right)L(1,0,-1)+\frac{1}{2}L(y,x,-\delta)=\frac{1}{2}. \tag{13}$$

Subtracting (12) from (13), we obtain

$$L(x,y,\delta) = L(1,0,-1)+(L(0,1,1)-L(1,0,-1))\left(\frac{1}{2}+\frac{y+\delta-x}{4}\right).$$

It follows that L is a linear function of $y+\delta-x$, and thus, from Corollary 2, L must satisfy $L(x,y,\delta) = \frac{1}{2}+\beta(y+\delta-x)$ for some real number β.

Finally, consider $x, y \in [0,1]$ and $\delta \in [-1,1]$ such that $y+\delta > x$ and the environment F such that $\mu_{\tau c}(x) = \mu_{\tau d}(y+\delta) = \mu_{\tau' d}(y) = \mu_{\tau\tau' d}(\delta) = 1$. Since $L_{cd}(\tau,\tau') = L(x,y,\delta) = \frac{1}{2}+\beta(y+\delta-x)$, payoff ordering implies that $\beta > 0$. ■

To close the proof of Proposition 1, since the range of L is $[0,1]$, $x, y \in [0,1]$, and $\delta \in [-1,1]$, we also need $\beta \leq \frac{1}{4}$.

8 Appendix B: Formal derivation of the dynamic system

We first introduce the probability space and individual space.

Probability space. The population is modeled as an index probability space $(W, \mathcal{W}, \lambda)$ where W is the continuum of individuals, $\mathcal{W}$ is a $\sigma-$algebra of subsets of W, and λ is a super-atomless measure; in particular $\lambda(A) = \alpha$ and $\lambda(B) = 1-\alpha$.[37] Time is indexed by $t \in \mathbb{R}^+$ with the Borel $\sigma-$algebra denoted by $\mathcal{B}$. The probability space that we use to model the random aspects of the imitation process is $(\Omega, \mathcal{F}, P)$.[38]

Dynamical system. Our analysis is concerned with the fraction of type A individuals choosing a, $p(\omega, t)$, and the fraction of type B individuals choosing b, $q(\omega, t)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}^+$. In order to analyze the paths $(p(\omega,t), q(\omega,t))_{t\in\mathbb{R}^+}$, we define the *state function* $\rho : W \times \Omega \times \mathbb{R}^+ \to \Sigma$ where $\rho(i,\omega,t)$ is the state of individual

[37] For a definition of super-atomless measures see Definition 5 and the appendix of Podczeck (2010).

[38] The analysis also assumes a right-continuous filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$ in the probability space $(\Omega, \mathcal{F}, P)$, such that all null events are included in $\mathcal{F}_0$; but this filtration will not appear (explicitly) in our analysis.

i in the state of the world ω, at time t. We also define the *sampling function* $\pi : W \times \Omega \times \mathbb{R}^+ \to W \cup \{J\}$ specifying the individual $\pi(i, \omega, t)$ that the individual i samples at time t in the state of the world ω, where $\pi(i, \omega, t) = J$ means that individual i does not sample any other individual at time t in the state of the world ω. We assume that $\pi(i, \omega, t) \neq J$ implies $\pi(\pi(i, \omega, t), \omega, t) = i$; that is, individuals sample each other.

The fraction of individuals whose state is σ at time t in the state of the world ω, denoted by $p_\sigma(\omega, t)$, is given by $p_\sigma(\omega, t) = \lambda(\{i \in W : \rho(i, \omega, t) = \sigma\})$ for all $\sigma \in \Sigma$, $\omega \in \Omega$, and $t \in \mathbb{R}^+$. Therefore, we have the following versions of (2) and (3) that express directly the state of the world dependence of the fractions of individuals making optimal choices:

$$\alpha p(\omega, t) = \sum_{\sigma \in \Sigma : \tau = A, c = a} p_\sigma(\omega, t) \tag{14}$$

and

$$(1 - \alpha) q(\omega, t) = \sum_{\sigma \in \Sigma : \tau = B, c = b} p_\sigma(\omega, t). \tag{15}$$

Differential equations of the system. We now provide the differential equations governing the fractions of the population making their optimal choices within each type.

Proposition 4 *Fix the initial fractions of the population in each individual state at* $(p_\sigma(0))_{\sigma \in \Sigma} \in \hat{\Delta}(\Sigma)$. *There exists a Fubini Extension* $(W \times \Omega, \mathcal{W} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ [39] *in which the state and sampling function* (ρ, π) *are defined, such that: (i)* $(p(\omega, t), q(\omega, t))$ *is deterministic almost surely; and (ii) for* P*−almost all* $\omega \in \Omega$, $(p(\omega, \cdot), q(\omega, \cdot)) : \mathbb{R}^+ \to [0, 1]^2$ *is the solution of the system of differential equations*

$$\begin{aligned} \alpha \dot{p} &= \alpha(1 - p)\left[\alpha p + (1 - \alpha)(1 - q)\right] L_{ba}(A) - \alpha p \left[\alpha(1 - p) + (1 - \alpha) q\right] L_{ab}(A) && (16) \\ (1 - \alpha)\dot{q} &= (1 - \alpha)(1 - q)\left[(1 - \alpha) q + \alpha(1 - p)\right] L_{ab}(B) - (1 - \alpha) q \left[(1 - \alpha)(1 - q) + \alpha p\right] L_{ba}(B), && (17) \end{aligned}$$

with initial condition $(p(\omega, 0), q(\omega, 0)) = \left(\alpha^{-1} \sum_{\sigma \in \Sigma : \tau = A, c = a} p_\sigma(0), (1 - \alpha)^{-1} \sum_{\sigma \in \Sigma : \tau = B, c = b} p_\sigma(0)\right)$. [40]

Before providing the proof of Proposition 4 we need to introduce the following definition, which is adapted from Duffie, Qiao and Sun (2016).

Definition 2 *Consider a pair of a state and a sampling function* (ρ, π) *defined on a Fubini Extension* $(W \times \Omega, \mathcal{W} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, $(p_\sigma(0))_{\sigma \in \Sigma} \in \hat{\Delta}(\Sigma)$, and $\left((\varsigma_{\sigma\sigma'}(\sigma''))_{\sigma'' \in \Sigma}\right)_{(\sigma, \sigma') \in \Sigma^2}$ *with* $(\varsigma_{\sigma\sigma'}(\sigma''))_{\sigma'' \in \Sigma} \in \Delta(\Sigma)$ *for all* $(\sigma, \sigma') \in$

[39] Theorem 1 in Podczeck (2010) shows that a sufficient condition allowing $\lambda \boxtimes P$ to be a rich Fubini extension (i.e., a proper extension of the product measure of λ and P on $W \times \Omega$ that allows for essentially pairwise independent random variables that are not essentially constant), is that the measure in the index probability space (λ) is super-atomless, which we have assumed from the outset.

[40] As in most of the equations below, the time and state-of-the-world dependence of p and q have been omitted.

Σ^2. *The pair* (ρ,π) *is said to be a* continuous time dynamical system with independent random sampling and independent random state-changing with parameters[41] $(p_\sigma(0))_{\sigma\in\Sigma}$ and $\left((\varsigma_{\sigma\sigma'}(\sigma''))_{\sigma''\in\Sigma}\right)_{(\sigma,\sigma')\in\Sigma^2}$ *(denoted by DS) if: (i)* $(p_\sigma(\omega,t))_{\sigma\in\Sigma}$ *is deterministic almost surely with given initial conditions* $(p_\sigma(\omega,0))_{\sigma\in\Sigma} = (p_\sigma(0))_{\sigma\in\Sigma}$*; (ii) for* $\lambda-$*almost every* $i\in W$*,* $\rho(i,\cdot,\cdot)$ *is a continuous time Markov chain in* Σ *with transition intensity*

$$R_{\sigma\sigma''}(\omega,t) = \sum_{\sigma'\in\Sigma} p_{\sigma'}(\omega,t)\varsigma_{\sigma\sigma'}(\sigma''), \tag{18}$$

for all two different states σ *and* σ'' *in* Σ*, and* $R_{\sigma\sigma}(\omega,t) = -\sum_{\sigma'\in\Sigma\setminus\{\sigma\}} R_{\sigma\sigma'}(\omega,t)$ *for all* $\sigma\in\Sigma$*,* $\omega\in\Omega$*, and* $t\in\mathbb{R}^+$*; (iii)* $\rho(i,\omega,t)$ *is* $(\mathcal{W}\boxtimes\mathcal{F})\otimes\mathcal{B}-$*measurable; and (iv) for* $\lambda-$*almost all* $i\in W$*,* $\rho(i,\cdot,t)$ *and* $\rho(j,\cdot,t)$ *are independent for* $\lambda-$*almost all* $j\in W$*.*

Now we are ready to provide the proof of Proposition 4.

Proof. From Corollary 2 in Duffie, Qiao and Sun (2016), there exists a Fubini Extension $(W\times\Omega,\mathcal{W}\boxtimes\mathcal{F},\lambda\boxtimes P)$ such that the pair of state and sampling function (ρ,π), defined on $(W\times\Omega,\mathcal{W}\boxtimes\mathcal{F},\lambda\boxtimes P)$, is a DS with parameters $(p_\sigma(0))_{\sigma\in\Sigma}\in\hat{\Delta}(\Sigma)$ and $\left((\varsigma_{\sigma\sigma'}(\sigma''))_{\sigma''\in\Sigma}\right)_{(\sigma,\sigma')\in\Sigma^2}$ with $(\varsigma_{\sigma\sigma'}(\sigma''))_{\sigma''\in\Sigma}\in\Delta(\Sigma)$ for all $(\sigma,\sigma')\in\Sigma^2$.[42] From their Corollary 1, for $P-$almost all ω, $(p_\sigma(\omega,t))_{\sigma\in\Sigma}$ is the solution of the system of differential equations

$$\dot{p}_\sigma(\omega,t) = \sum_{\sigma''\in\Sigma\setminus\{\sigma\}} p_{\sigma''}(\omega,t)\sum_{\sigma'\in\Sigma} p_{\sigma'}(\omega,t)\varsigma_{\sigma''\sigma'}(\sigma) - p_\sigma(\omega,t)\sum_{\sigma''\in\Sigma\setminus\{\sigma\}}\sum_{\sigma'\in\Sigma} p_{\sigma'}(\omega,t)\varsigma_{\sigma\sigma'}(\sigma'') \tag{19}$$

for all $\sigma\in\Sigma$, with $(p_\sigma(\omega,0))_{\sigma\in\Sigma} = (p_\sigma(0))_{\sigma\in\Sigma}$. Therefore,[43]

$$\begin{aligned}\alpha\dot{p} &= \sum_{\sigma\in\Sigma:\tau=A,c=a}\dot{p}_\sigma \\ &= \sum_{\sigma\in\Sigma:\tau=A,c=a}\sum_{\sigma''\in\Sigma\setminus\{\sigma\}} p_{\sigma''}\sum_{\sigma'\in\Sigma} p_{\sigma'}\varsigma_{\sigma''\sigma'}(\sigma) - \sum_{\sigma\in\Sigma:\tau=A,c=a} p_\sigma\sum_{\sigma''\in\Sigma\setminus\{\sigma\}}\sum_{\sigma'\in\Sigma} p_{\sigma'}\varsigma_{\sigma\sigma'}(\sigma'')\end{aligned} \tag{20}$$

If $\tau = A$ and $\tau'' = B$, then $\varsigma_{\sigma''\sigma'}(\sigma) = \varsigma_{\sigma\sigma'}(\sigma'') = 0$ for all $\sigma'\in\Sigma$. Therefore, in both the minuend and

[41] In the definition in Duffie, Qiao and Sun (2016), dynamical systems also depend on parameters describing mutation intensities and matching intensities. In our model there are no mutations so these parameters are all zero and hence we omit them, and the matching intensities of our model are given by the fraction of the population of the state of the individual to be sampled; thus we omit these parameters as well. We only consider different intensities in Section 4 and Appendix D.

[42] Corollary 2 in Duffie, Qiao and Sun (2016) considers a slightly more general formulation of Definiton 2 (that, as mentioned in footnote 41, allows for exogenous mutation rates of individual states and matching intensities functions where matching intensities of individuals of different states are not necessarily proportional to the fraction of the population of the individual state of the sampled individual). For that more general formulation, this corollary establishes the existence of a Fubini Extension where a DS (ρ,π) can be defined. Their Corollary 1, which we use below, provides the differential equations describing the path of the expected value of the fraction of individuals in each individual state and establishes that the realization of this path is almost surely equal to its expected value.

[43] In the rest of the proof we omit the time and state-of-the-world dependence of p_σ, p, and q, and their time derivatives.

subtrahend, the second summation is over $\{\sigma'' \in \Sigma : \tau'' = A\}$; thus,

$$\begin{aligned}
\alpha \dot{p} &= \sum_{\sigma \in \Sigma : \tau = A, c = a} \sum_{\sigma'' \in \Sigma : \tau'' = A} p_{\sigma''} \sum_{\sigma' \in \Sigma} p_{\sigma'} \varsigma_{\sigma''\sigma'}(\sigma) - \sum_{\sigma \in \Sigma : \tau = A, c = a} p_{\sigma} \sum_{\sigma'' \in \Sigma : \tau'' = A} \sum_{\sigma' \in \Sigma} p_{\sigma'} \varsigma_{\sigma\sigma'}(\sigma'') \\
&= \sum_{\sigma \in \Sigma : \tau = A, c = a} \sum_{\sigma'' \in \Sigma : \tau'' = A, c'' = b} p_{\sigma''} \sum_{\sigma' \in \Sigma} p_{\sigma'} \varsigma_{\sigma''\sigma'}(\sigma) - \sum_{\sigma \in \Sigma : \tau = A, c = a} p_{\sigma} \sum_{\sigma'' \in \Sigma : \tau'' = A, c'' = b} \sum_{\sigma' \in \Sigma} p_{\sigma'} \varsigma_{\sigma\sigma'}(\sigma''),
\end{aligned}$$

where the second equality is obtained canceling the terms appearing in the minuend and subtrahend out. Next, since $c = a$ and $c'' = b$ we have $\varsigma_{\sigma''\sigma'}(\sigma) = 0$ whenever $c' = b$ and $\varsigma_{\sigma\sigma'}(\sigma'') = 0$ whenever $c' = a$. Therefore, in the minuend, the third summation is over $\{\sigma' \in \Sigma : c' = a\}$ and, in the subtrahend, the third summation is over $\{\sigma' \in \Sigma : c' = b\}$. Thus,

$$\alpha \dot{p} = \sum_{\sigma \in \Sigma : \tau = A, c = a} \sum_{\sigma'' : \tau'' = A, c'' = b} p_{\sigma''} \sum_{\sigma' \in \Sigma : c' = a} p_{\sigma'} \varsigma_{\sigma''\sigma'}(\sigma) - \sum_{\sigma \in \Sigma : \tau = A, c = a} p_{\sigma} \sum_{\sigma'' \in \Sigma : \tau'' = A, c'' = b} \sum_{\sigma' \in \Sigma : c' = b} p_{\sigma'} \varsigma_{\sigma\sigma'}(\sigma''). \quad (21)$$

The minuend in the right-hand-side of (21), denoted by M, may be written as

$$\begin{aligned}
M &= \sum_{\sigma'' \in \Sigma : \tau'' = A, c'' = b} \sum_{\sigma' \in \Sigma : c' = a} p_{\sigma''} p_{\sigma'} \left(\sum_{\sigma : \tau = A, c = a} \varsigma_{\sigma''\sigma'}(\sigma) \right) \\
&= \sum_{\sigma'' \in \Sigma : \tau'' = A, c'' = b} \sum_{\sigma' \in \Sigma : c' = a} p_{\sigma''} p_{\sigma'} L(x'', x', \delta''_{\tau'}),
\end{aligned}$$

where we have used the definition of $\left((\varsigma_{\sigma\sigma'}(\sigma''))_{\sigma'' \in \Sigma} \right)_{(\sigma, \sigma') \in \Sigma^2}$ in (1) and the fact that

$$\sum_{\sigma : \tau = A, c = a} \mu_{Aa}(x) \mu_{AAb}(\delta_A) \mu_{ABb}(\delta_B) = 1.$$

Therefore,

$$\begin{aligned}
M &= \sum_{\sigma'' \in \Sigma : \tau = A, c = b} \left(\sum_{\sigma' \in \Sigma : \tau' = A, c' = a} p_{\sigma''} p_{\sigma'} L(x'', x', \delta''_A) + \sum_{\sigma' \in \Sigma : \tau' = B, c' = a} p_{\sigma''} p_{\sigma'} L(x'', x', \delta''_B) \right) \\
&= \alpha(1-p)\alpha p \sum_{x'' \in X, \delta''_A \in \Delta} \left(\sum_{x' \in X} \mu_{Ab}(x'') \mu_{Aa}(x') \mu_{AAa}(\delta''_A) L(x'', x', \delta''_A) \right) \\
&\quad + \alpha(1-p)(1-\alpha)(1-q) \sum_{x'' \in X, \delta''_B \in \Delta} \left(\sum_{x' \in X} \mu_{Ab}(x'') \mu_{Ba}(x') \mu_{ABa}(\delta''_B) L(x'', x', \delta''_B) \right) \\
&= \alpha(1-p) \left[\alpha p + (1-\alpha)(1-q) \right] L_{ba}(A),
\end{aligned}$$

where the second equality follows from the fact that

1. $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} p_\sigma = \alpha p \mu_{Aa}(x_0)$,

2. $\sum_{\sigma\in\Sigma:\tau=A,c=b,x=x_0,\delta_A=\delta} p_\sigma = \alpha(1-p)\mu_{Ab}(x_0)\mu_{AAa}(\delta)$,

3. $\sum_{\sigma\in\Sigma:\tau=B,c=a,x=x_0} p_\sigma = (1-\alpha)(1-q)\mu_{Ba}(x_0)$, and

4. $\sum_{\sigma\in\Sigma:\tau=A,c=b,x=x_0,\delta_B=\delta} p_\sigma = \alpha(1-p)\mu_{Ab}(x_0)\mu_{ABa}(\delta)$,

for all $x_0 \in X$ and $\delta \in \Delta$.[44]

Similarly, the subtrahend of (21) can be written as

$$-\sum_{\sigma\in\Sigma:\tau=A,c=a} p_\sigma \sum_{\sigma''\in\Sigma:\tau''=A,c''=b} \sum_{\sigma'\in\Sigma:c'=b} p_{\sigma'}\varsigma_{\sigma\sigma'}(\sigma'') = -\alpha p\left[\alpha(1-p)+(1-\alpha)q\right]L_{ab}(A)$$

thus,

$$\alpha\dot{p} = \alpha(1-p)\left[\alpha p+(1-\alpha)(1-q)\right]L_{ba}(A) - \alpha p\left[\alpha(1-p)+(1-\alpha)q\right]L_{ab}(A).$$

The analogous argument yields (17). ■

9 Appendix C: Proof of Theorem 1

Proof. First we establish asymptotic stability. Define the Jacobian matrix of $(\dot{p}, \dot{q})$

$$J(p,q) := \begin{bmatrix} \dot{p}_1(p,q) & \dot{p}_2(p,q) \\ \dot{q}_1(p,q) & \dot{q}_2(p,q) \end{bmatrix},$$

where $\dot{p}_i(p,q)$ and $\dot{q}_i(p,q)$ denote the corresponding partial derivatives of $\dot{p}(p,q)$ and $\dot{q}(p,q)$ with respect to their i^{th} arguments, i.e., with respect to p or q. A rest point (p^*,q^*) is asymptotically stable if the real part of the eigenvalues of $J(p^*,q^*)$ are negative (see, e.g., Sydsaeter, Hammond, Seierstad and Strom 2008, Theorems 6.8.1 and 7.5.1). This is equivalent to $Det(J(p^*,q^*)) > 0$ and $Tr(J(p^*,q^*)) < 0$, where $Det(J(p^*,q^*))$ and $Tr(J(p^*,q^*))$ are the determinant and trace of $J(p^*,q^*)$, respectively. Consider first $(1,0)$. We have $Tr(J(1,0)) = 2D-(1+U)-\alpha(U+D-1) < 0$. Next, $Det(J(1,0)) = U(1-2D)+\alpha(U+D-1) > 0$ is

[44] To see why 1. holds observe that, from (19), we have $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} \dot{p}_\sigma$ is equal to $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0}\sum_{\sigma''\in\Sigma\setminus\{\sigma\}} p_{\sigma''}\sum_{\sigma'\in\Sigma} p_{\sigma'}\varsigma_{\sigma''\sigma'}(\sigma) - \sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} p_\sigma \sum_{\sigma''\in\Sigma\setminus\{\sigma\}}\sum_{\sigma'\in\Sigma} p_{\sigma'}\varsigma_{\sigma\sigma'}(\sigma'')$. From (1), the minuend is $\mu_{Aa}(x_0)$ times the minuend in (20). The initial conditions imply that, at time $t=0$, the subtrahend is $\mu_{Aa}(x_0)$ times the subtrahend in (20). Thus, at $t=0$, $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} \dot{p}_\sigma = \mu_{Aa}(x_0)\alpha\dot{p}$. Therefore, at $t=0$, the right derivative of $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} p_\sigma/(\alpha p)$ with respect to t is 0. Furthermore, the derivative of $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} p_\sigma/(\alpha p)$ with respect to t is negative (positive) when $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} p_\sigma > (<)\mu_{Aa}(x_0)$. Thus, $\sum_{\sigma\in\Sigma:\tau=A,c=a,x=x_0} p_\sigma/(\alpha p)$ is constant over time and equal to its value at $t=0$, $\mu_{Aa}(x_0)$. The proof of the facts 2.-4. is similar and it is omitted.

equivalent to $\alpha > \frac{U(1-2D)}{U+D-1} = \overline{\alpha}$. An analogous calculation holds for $(0,1)$. Now, consider $(\hat{p}, \hat{q})$. Note that

$$Tr(J(\hat{p}, \hat{q})) = \frac{2U(1-U)(\alpha + (1-2D)^2) + 2D(1-D)(2-\alpha) - 1}{(2D-1)(2U-1)} < 0.$$

Next,

$$Det(J(\hat{p}, \hat{q})) = \frac{(\alpha(U+D-1) - (1-U)(2D-1))(U(2D-1) - \alpha(U+D-1))}{(2U-1)(2D-1)} > 0$$

if $\alpha > \frac{(1-U)(2D-1)}{(U+D-1)} = \underline{\alpha}$ and $\alpha < \frac{U(2D-1)}{(U+D-1)} = \overline{\alpha}$.

In order to prove that the asymptotically stable points are global attractors, notice that $\dot{p}_2, \dot{q}_1 < 0$, hence $\lim_{t \to \infty}(p(t), q(t)) \in RP$ for all paths (see, e.g., Theorem 3.4.1 in Hofbauer and Sigmund, 1998). It is easy to verify that all $(p,q) \in RP \setminus \{(p^*, q^*)\}$ are saddle-path stable with no stable arm in $[0,1]^2$. Hence, the system always converges to the asymptotically stable point. ■

10 Appendix D: Proofs Section 4

10.1 Dynamical systems with biased sampling

In this subsection we briefly sketch how to derive (10)-(11). The general version of equation (18), when we allow for general intensities, is

$$R_{\sigma\sigma''}(\omega, t) = \sum_{\sigma' \in \Sigma} \theta_{\sigma\sigma'}((p_{\sigma'''}(\omega, t))_{\sigma''' \in \Sigma}) \varsigma_{\sigma\sigma'}(\sigma'')$$

and the corresponding version of equation (19) is[45]

$$\dot{p}_\sigma = \sum_{\sigma'' \in \Sigma \setminus \{\sigma\}} p_{\sigma''} \sum_{\sigma' \in \Sigma} \theta_{\sigma''\sigma'}((p_{\sigma'''})_{\sigma''' \in \Sigma}) \varsigma_{\sigma''\sigma'}(\sigma) - p_\sigma \sum_{\sigma'' \in \Sigma \setminus \{\sigma\}} \sum_{\sigma' \in \Sigma} \theta_{\sigma\sigma'}((p_{\sigma'''})_{\sigma''' \in \Sigma}) \varsigma_{\sigma\sigma'}(\sigma''). \quad (22)$$

Then, (22) and analogous derivations to those following equation (19) in Appendix B yield (10)-(11).

10.2 Proofs of Stable equilibria

First we provide the proof of Lemma 2.

Proof. (of Lemma 2) (i) We use the determinant and trace of the Jacobian matrix of the system (4)-(5). Recall that (p,q) is asymptotically stable if and only if $Det(J(p,q)) > 0$ and $Tr(J(p,q)) < 0$. Now, $Det(J(1,0)) =$

[45]Dependence on time and state of the world are omitted in the notation.

$U-D+\alpha_B D(1-2U)+\alpha_A(2D-1)(1-U)) > 0$ if and only if $\alpha_A > \overline{\alpha}_A(\alpha_B)$. $\overline{\alpha}_A(\alpha_B) \geq 1$ for all $\alpha_B \geq \frac{1-D}{D}$, so if $\alpha_A > \overline{\alpha}_A(\alpha_B)$, then $\alpha_B < \frac{1-D}{D}$. Finally, if $\alpha_B < \frac{1-D}{D}$, then $Tr(J(1,0)) = \alpha_A - 1 - U(1+\alpha_A) + D(1+\alpha_B) < 0$. (ii) is established analogously. (iii) follows from (i) and (ii) and the fact that $\overline{\alpha}_A(\alpha_B) > \left[\overline{\alpha}_B^{-1}\right](\alpha_B)$ for all $\alpha_B \in (0,1)$. ■

We now provide the proof of Corollary 1.

Proof. (of Corollary 1) Part (i) follows by observing that $\overline{\alpha}_A(\alpha_B) > 1$ for all $\alpha_B > \frac{1-D}{D}$, and if $U > D$, then $\overline{\alpha}_A(\alpha_B) < 0$ for all $\alpha_B < \frac{U-D}{D(2U-1)}$. An analogous argument proves (ii). ■

Now we provide the results that lead to the proof of Lemma 3. Define $[\overline{p}^{-1}] : [0,1] \to [0,1]$ with $[\overline{p}^{-1}](p) := \max\{q : \overline{p}(q) = p\}$ for all $p \in [0,1]$. Define $[\overline{q}^{-1}] : [0,1] \to [0,1]$ with $[\overline{q}^{-1}](q) = \max\{p : \overline{q}(p) = q\}$ for all $q \in [0,1]$. Notice that $\overline{p}$ is strictly decreasing on $\{q \in [0,1] : \overline{p}(q) < 1\}$. To see this, note that the concavity of $\overline{p}$ and $\overline{p}(0) = 1$ together imply that for all $q \in (0,1]$ such that $\overline{p}(q) < 1$ and $\lambda \in (0,1]$ we have $\overline{p}((1-\lambda)q) = \overline{p}(\lambda 0 + (1-\lambda)q) \geq \lambda 1 + (1-\lambda)\overline{p}(q) > \overline{p}(q)$. Combining with the fact that $\overline{p}(1) = 0$, it follows that $\overline{p}$ is strictly decreasing on $[w,1]$ and there is no $q \in [0,w]$ with $\overline{p}(q) \in [0,1)$, where $w := [\overline{p}^{-1}](1)$. Therefore, $[\overline{p}^{-1}]$ is the well defined real-valued inverse function of $\overline{p}$ in the restricted domain $[w,1]$ (with range $[0,1]$), which is a continuous, decreasing, and concave function. An analogous argument holds for $\overline{q}$ and $[\overline{q}^{-1}]$.

Remark 4 *(i) $\alpha_A > (<)\overline{\alpha}_A(\alpha_B)$ if and only if $\overline{p}'(0) < (>)[\overline{q}^{-1}]'(0)$. (ii) $\alpha_B > (<)\overline{\alpha}_B(\alpha_A)$ if and only if $\overline{q}'(0) < (>)[\overline{p}^{-1}]'(0)$.*

Proof. (i) Notice that $[\overline{q}^{-1}]'(0) = \frac{1}{\overline{q}'(\overline{q}^{-1}(0))} = \frac{1}{\overline{q}'(1)}$. Next, $\overline{p}'(0) = \frac{\alpha_B D+D-1}{D(1-\alpha_B)}$ and $\frac{1}{\overline{q}'(1)} = \frac{(1-U)(1-\alpha_A)}{\alpha_A(1-U)-U}$ and $\frac{\alpha_B D+D-1}{D(1-\alpha_B)} < \frac{(1-U)(1-\alpha_A)}{\alpha_A(1-U)-U}$ can be written $\alpha_A > \frac{U-D+\alpha_B D(1-2U)}{(1-U)(1-2D)} = \overline{\alpha}_A(\alpha_B)$. Analogous calculations hold for (ii). ■

In the sequel, unless stated otherwise, the domains of $\overline{p}$ and $\overline{q}$ are the whole set of real numbers. The following results follow from straightforward calculus:

Remark 5 *$\overline{q}(\overline{p}(q)) - q = 0$ is a polynomial equation of degree 4, and consequently has at most 4 different solutions in $q \in \mathbb{R}$.*

Remark 6 *$\overline{q}(p)$ has the following properties: (i) $\overline{q}(0) = 1$ and $\overline{q}(1) = 0$, (ii) $\lim_{p\to\infty} \overline{q}(p) = -\lim_{p\to-\infty} \overline{q}(p) = \lim_{p\to\infty}\left(\frac{\alpha_A(2U-1)-U}{(1-\alpha_A)(1-2U)} + \frac{\alpha_A}{1-\alpha_A}p\right)$, (iii) $\overline{q}$ is discontinuous only at $\frac{U}{2U-1}$. In particular $\lim_{p\to(\frac{U}{2U-1})^-} \overline{q}(p) = -\infty$ and $\lim_{p\to(\frac{U}{2U-1})^+} \overline{q}(p) = \infty$ and (iv) $\overline{q}$ has two local extrema, a local maximum at some $p < 1$ and a local minimum at some $p > U/(2U-1)$.*

The following Lemma shows that one of the four roots of $\overline{q}(\overline{p}(q)) - q = 0$ is located outside of $[0,1]$, which implies that if there is an interior rest point it is unique.

Lemma 7 *If $\alpha_A \neq 1-\alpha_B$, then there is some $z \notin [0,1]$ such that $\overline{q}(\overline{p}(z)) - z = 0$ with $\overline{p}(z) \in (-\infty, 0) \cup (\frac{U}{2U-1}, \infty)$.*

Proof. (i) Consider the case $\frac{\alpha_A}{1-\alpha_A} < \frac{1-\alpha_B}{\alpha_B}$. Consider $\overline{q}$ with its domain restricted to $(\frac{U}{2U-1}, \infty)$ and $\overline{p}$ with its domain restricted to $(\frac{D}{2D-1}, \infty)$. Both $\overline{q}$ and $\overline{p}$ are continuous on $(\frac{U}{2U-1}, \infty)$ and $(\frac{D}{2D-1}, \infty)$, respectively. Since $\lim_{p\to\infty} \overline{q}(p) = \infty$ and $\lim_{q\to(\frac{D}{2D-1})^+} \overline{p}(q) = \infty$ there is a point $q' \in (\frac{D}{2D-1}, \infty)$ (close to $\frac{D}{2D-1}$) such that $q' < \overline{q}(\overline{p}(q'))$, which means that $(\overline{p}(q'), q')$ is in the subgraph of $\overline{q}(p)$. Next, given that $\frac{\alpha_A}{1-\alpha_A} < \frac{1-\alpha_B}{\alpha_B}$ there is some $q'' \in (\frac{D}{2D-1}, \infty)$ (sufficiently large) such that $\overline{p}(q'') \simeq \frac{\alpha_B(2D-1)-D}{(1-\alpha_B)(1-2D)} + \frac{\alpha_B}{1-\alpha_B} q''$, $\overline{q}(\overline{p}(q'')) \simeq \frac{\alpha_A(2U-1)-U}{(1-\alpha_A)(1-2U)} + \frac{\alpha_A}{1-\alpha_A}\overline{p}(q'')$ and $q'' \simeq -\frac{(1-\alpha_B)(1-2D)}{\alpha_B(2D-1)-D}\frac{1-\alpha_B}{\alpha_B} + \frac{1-\alpha_B}{\alpha_B}\overline{p}(q'') > \overline{q}(\overline{p}(q''))$. This means that $(\overline{p}(q''), q'')$ is in the epigraph of $\overline{q}(p)$. From Remark 6 and an analogous analysis for $\overline{p}$, we have that $\overline{p}$ is continuous on $(\frac{D}{2D-1}, \infty)$, $\overline{q}$ is continuous on $(\frac{U}{2U-1}, \infty)$ and $\lim_{p\to\infty} \overline{q}(p) = \lim_{p\to(\frac{U}{2U-1})^+} \overline{q}(p) = \infty$. Since $(\overline{p}(q'), q')$ is in the subgraph of $\overline{q}$ and $(\overline{p}(q''), q'')$ is in the epigraph of $\overline{q}(p)$, with $q', q'' \in (\frac{D}{2D-1}, \infty)$, there is a point $(\overline{p}(z), z)$ with $z \in (\frac{D}{2D-1}, \infty)$ that is both in the subgraph and in the epigraph of $\overline{q}$. Hence $\overline{q}(\overline{p}(z)) = z$ for some $z \notin [0,1]$, such that $\overline{p}(z) \in (\frac{U}{2U-1}, \infty)$.

Consider $\frac{\alpha_A}{1-\alpha_A} > \frac{1-\alpha_B}{\alpha_B}$. Consider $\overline{q}$ and $\overline{p}$ with their domains restricted to $[0, -\infty)$. Both $\overline{q}$ and $\overline{p}$ are continuous on $[0, -\infty)$. The point $(\overline{p}(q'), q')$, with $q' \in [0, -\infty)$ such that $\overline{p}(q') = 0$ is in the subgraph of $\overline{q}(p)$, because $\overline{q}(\overline{p}(q')) = 1 > q'$. Next, given that $\frac{\alpha_A}{1-\alpha_A} > \frac{1-\alpha_B}{\alpha_B}$ there is some $q'' \in (0, -\infty)$ (sufficiently small) such that $\overline{p}(q'') \simeq \frac{\alpha_B(2D-1)-D}{(1-\alpha_B)(1-2D)} + \frac{\alpha_B}{1-\alpha_B} q''$, $\overline{q}(\overline{p}(q'')) \simeq \frac{\alpha_A(2U-1)-U}{(1-\alpha_A)(1-2U)} + \frac{\alpha_A}{1-\alpha_A}\overline{p}(q'')$ and $q'' \simeq -\frac{(1-\alpha_B)(1-2D)}{\alpha_B(2D-1)-D}\frac{1-\alpha_B}{\alpha_B} + \frac{1-\alpha_B}{\alpha_B}\overline{p}(q'') > \overline{q}(\overline{p}(q''))$. This means that $(\overline{p}(q''), q'')$ is in the epigraph of $\overline{q}$. Since $\overline{p}$ and $\overline{q}$ are continuous on $[0, \infty)$, and since $(\overline{p}(q'), q')$ is in the subgraph of $\overline{q}$ and $(\overline{p}(q''), q'')$ is in the epigraph of $\overline{q}$, there must be a point $(\overline{p}(z), z)$, with $z \in (-\infty, 0)$, that is both in the subgraph and in the epigraph of $\overline{q}$. Hence $\overline{q}(\overline{p}(z)) = z$ for some $z \in (-\infty, 0)$ at which $\overline{p}(z) \in (-\infty, 0)$.

Hence, $\overline{q}(\overline{p}(z)) - z = 0$ for some $z \in (-\infty, 0) \cup (\frac{D}{2D-1}, \infty)$. ■

Now we provide the proof of Lemma 3.

Proof. (of Lemma 3) (i) Suppose $\alpha_A > \overline{\alpha}_A(\alpha_B)$. Then by Remark 4 $\overline{p}'(0) < [\overline{q}^{-1}]'(0)$. Define $[\widetilde{q}^{-1}] : (-\infty, 0] \to \mathbb{R}$ with $[\widetilde{q}^{-1}](q) := \{p : \overline{q}(p) = q\}$ for $q \in (-\infty, 0]$. Both $[\widetilde{q}^{-1}]$ and $\overline{p}$ are continuous on $(-\infty, 0)$. Since $\overline{p}'(0) < [\widetilde{q}^{-1}]'(0)$ there is some $q' < 0$ (close to 0) such that $\overline{p}(q') > [\widetilde{q}^{-1}](q')$. Next, $\lim_{q\to\infty} \overline{p}(q) = -\infty$ while $\lim_{q\to\infty} [\widetilde{q}^{-1}](q) = \frac{U}{2U-1}$. This means that there is some sufficiently large q'' such that $\overline{p}(q'') > [\widetilde{q}^{-1}](q'')$. Hence, there is some $z \in (-\infty, 0)$ such that $\overline{p}(z) = [\widetilde{q}^{-1}](z)$ and $1 < \overline{p}(z) < \frac{U}{2U-1}$. Since there are at most four solutions to $\overline{q}(\overline{p}(p)) - p = 0$ and there is one with $\overline{p}(q) \in (-\infty, 0) \cup (\frac{U}{2U-1}, \infty)$ and one such that $\overline{p}(q) \in (1, \frac{U}{2U-1})$ there is no solution in $(0, 1)$. An analogous argument holds if $\alpha_B > \overline{\alpha}_B(\alpha_A)$.

(ii) Suppose. $\alpha_A < \overline{\alpha}_A(\alpha_B)$ or $\alpha_B < \overline{\alpha}_B(\alpha_A)$. By Remark 4 $\overline{q}'(0) > [\overline{p}^{-1}]'(0)$, which means that there is some $p' \in (0, 1)$ (close to 0) such that $\overline{q}(p') > [\overline{p}^{-1}](p')$. By Remark 4 $\overline{p}'(0) > [\overline{q}^{-1}]'(0)$, which means that there is some $p'' \in (0, 1)$ (close to 1) such that $\overline{q}(p'') < [\overline{p}^{-1}](p'')$. Since $[\overline{p}^{-1}](p)$ and $\overline{q}(p)$ are continuous on $(0, 1)$

there is some $z \in (0,1)$ such that $[\overline{p}^{-1}](z) = \overline{q}(z)$ and hence there is an interior rest point. ■

If $\alpha_A = 1 - \alpha_B$ the expression for the interior rest point is given by $(\widetilde{p}, \widetilde{q}) = (\widehat{p}, \widehat{q})$. The following result provides the expression for the interior rest point $(\widetilde{p}, \widetilde{q})$ when $\alpha_A \neq 1 - \alpha_B$.

Lemma 8 *Suppose $\alpha_A \neq 1 - \alpha_B$. Then,*

$$\begin{aligned} \widetilde{p} &= \frac{(\alpha_A + \alpha_B - 1)(U - D + 2\alpha_A U(2D-1)) + \alpha_A\alpha_B(1 - U - D))}{2\alpha_A(\alpha_A + \alpha_B - 1)(2U-1)(2D-1)} \\ &+ \frac{\sqrt{\begin{array}{c} 4\alpha_A(\alpha_A+\alpha_B-1)(2D-1)(1-\alpha_A)U\left[U(1-\alpha_A) - \alpha_B(2U-1)(1-D) - D + 2\alpha_A UD\right] \\ + \left[(\alpha_A+\alpha_B-1)(U - D + 2\alpha_A U(2D-1)) + \alpha_A\alpha_B(1-U-D))\right]^2 \end{array}}}{2\alpha_A(\alpha_A + \alpha_B - 1)(2U-1)(2D-1)} \end{aligned}$$

The expression for $\widetilde{q}$ is analogous.

Proof. The terms p and $1-p$ can be factored out from the left hand side of the fourth degree polynomial $\overline{p}(\overline{q}(p)) - p = 0$. It is thus obtained that any interior rest point must be a solution to the second degree polynomial

$$\begin{gathered} \alpha_A(\alpha_A + \alpha_B - 1)(1-2U)^2(2D-1)p^2 \\ -(2U-1)\left[(\alpha_A+\alpha_B-1)(U - D + 2\alpha_A U(2D-1)) + \alpha_A\alpha_B(1-U-D)\right]p \\ -(1-\alpha_A)U\left[U(1-\alpha_A) - \alpha_B(2U-1)(1-D) - D + 2\alpha_A UD\right] = 0 \end{gathered}$$

The expression in the statement of the lemma is obtained by applying the quadratic formula and observing that $\widetilde{p} \in (0,1)$ is only consistent with the positive square root. ■

We now provide the proof of Proposition 2.

Proof. (of Proposition 2) Since $\dot{p}_2(p,q), \dot{q}_1(p,q) < 0$ the system always converges to a rest point as $t \to \infty$ (see, e.g., Theorem 3.4.1 in Hofbauer and Sigmund, 1998). If $\alpha_A > \overline{\alpha}_A(\alpha_B)$, then, by Lemma 3, $RP = \{(1,0),(0,1)\}$. By Lemma 2, $(0,1)$ is not asymptotically stable. Furthermore, $(0,1)$ has no stable arm in $[0,1]^2$. Hence, the system converges to $(1,0)$. Analogously if $\alpha_A < \left[\overline{\alpha}_B^{-1}\right](\alpha_B)$, then the system converges to $(0,1)$. If $\left[\overline{\alpha}_B^{-1}\right](\alpha_B) < \alpha_A < \overline{\alpha}_A(\alpha_B)$, then by Lemma 3, $RP = \{(1,0),(0,1),(\widetilde{p},\widetilde{q})\}$, for some $(\widetilde{p},\widetilde{q}) \in (0,1)^2$. By Lemma 2, $(1,0)$ and $(0,1)$ are not asymptotically stable and, furthermore, have no stable arm in $[0,1]^2$. Hence, the system converges to $(\widetilde{p},\widetilde{q})$. ■

10.3 Proofs for comparative statics of Section 4

Remark 7 *(i) $\widetilde{p}_1(\alpha_A,\alpha_B) > 0$, (ii) $\widetilde{p}_2(\alpha_A,\alpha_B) < 0$, (iii) $\widetilde{q}_1(\alpha_A,\alpha_B) > 0$ and (iv) $\widetilde{q}_2(\alpha_A,\alpha_B) < 0$.*

Proof. $\widetilde{q}(\alpha_A,\alpha_B)$ is defined by $\overline{q}(\overline{p}(\widetilde{q}(\alpha_A,\alpha_B))) - \widetilde{q}(\alpha_A,\alpha_B) = 0$. To establish (iv), we differentiate implicitly and obtain

$$\frac{\partial \widetilde{q}}{\partial \alpha_B} = -\frac{\overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B)))\frac{\partial \overline{p}(\widetilde{q}(\alpha_A,\alpha_B))}{\partial \alpha_B}}{\overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B))\overline{p}'(\widetilde{q}(\alpha_A, \alpha_B)) - 1}.$$

Consider first the denominator. It holds that $[\overline{p}^{-1}]'(\widetilde{p}(\alpha_A, \alpha_B)) > \overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B)))$. This means $\frac{1}{\overline{p}'(\widetilde{q}(\alpha_A,\alpha_B))} > \overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B)))$, or $1 < \overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B)))\overline{p}'(\widetilde{q}(\alpha_A, \alpha_B))$. Therefore, the denominator is positive. Now consider the denominator. Note that $\frac{\partial \overline{p}(\widetilde{q}(\alpha_A,\alpha_B))}{\partial \alpha_B} > 0$ and $\overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B))) < 0$. Hence, the numerator is negative, so $\frac{\partial \widetilde{q}(\alpha_A,\alpha_B)}{\partial \alpha_B} > 0$. An analogous procedure yields

$$\frac{\partial \widetilde{q}(\alpha_A, \alpha_B)}{\partial \alpha_A} = -\frac{\frac{\partial \overline{q}(\overline{p}(\widetilde{q}(\alpha_A,\alpha_B)))}{\partial \alpha_A}}{\overline{q}'(\overline{p}(\widetilde{q}(\alpha_A, \alpha_B))\overline{p}'(\widetilde{q}(\alpha_A, \alpha_B)) - 1} < 0.$$

Analogous arguments yield (i) and (ii). ■

We now provide the proof of Remark 3.

Proof. (of Remark 3) Consider a large enough α_A such that there is some $p' \in (0, 1)$ with $\dot{p}(p', 1) = 0$. Then, since $\overline{q}$ is concave and its graph contains $(p', 1)$ and $(1, 0)$ it holds that $\widetilde{p}(\alpha_A, \alpha_B) > p'$. Next, $\dot{p}(p', 1) = \alpha_A p'(1 - p')(2U - 1) - (1 - \alpha_A)p'(U - 1) = 0$ implies $p' = \frac{\alpha_A U+U-1}{\alpha_A(2U-1)}$, which approaches 1 as $\alpha_A \to 1$. Since $\widetilde{p}(\alpha_A, \alpha_B) > p'$, we obtain that $\lim_{\alpha_A\to 1} \widetilde{p}(\alpha_A, \alpha_B) = 1$. An analogous argument holds for $\widetilde{q}(\alpha_A, \alpha_B)$. ■

11 Appendix E: Proofs of Section 5

Proof of Proposition 3. Since $G(r) = 0$ for all $r \geq \varphi$ and G is continuous on $[0, \varphi]$, it attains its maximum somewhere in this interval. Both Q and G are twice continuously differentiable on $(\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$. Let $G'(x^+)$ and $G'(x^-)$ (corresp. $Q'(x^+)$ and $Q'(x^-)$) denote the right and left derivatives, respectively, of G (corresp. Q) for all $x \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$.

Suppose $k \geq \overline{r}(\varphi, \alpha)$. Then $G(r) < 0$ for all $r \in [0, \overline{r}(\varphi, \alpha))$ and $G(r) = 0$ for all $r \geq \overline{r}(\varphi, \alpha)$. Hence, in this case, the monopolist withdraws from the market.

Suppose $k < \overline{r}(\varphi, \alpha)$. Then

$$G'(\overline{r}(\varphi, \alpha)^-) = Q(\overline{r}(\varphi, \alpha)) + (\overline{r}(\varphi, \alpha) - k)Q'(\overline{r}(\varphi, \alpha)^-) = (\overline{r}(\varphi, \alpha) - k)Q'(\overline{r}(\varphi, \alpha)^-) < 0.$$

Hence, in this case $r^* < \overline{r}(\varphi, \alpha)$. From Lemma 4, $G'(\underline{r}(\varphi, \alpha)^+) > 0$ if $k > \hat{k}$. Hence, if $k > \hat{k}$, then $r^* > \underline{r}(\varphi, \alpha)$. This proves that $r^* \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$ if $k \in (\hat{k}, \overline{r}(\varphi, \alpha))$. In order to see that in this case r^* is the unique solution to $G'(r) = 0$, first note that $G(r) < 0$ for $r < k$ and hence, $r^* \geq k$. Next, for any k, r such that

$0 \leq k \leq r \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$ we have that

$$G''(r) = 2Q'(r) + (r - k)Q''(r) < 0.$$

If $Q''(r) \leq 0$ this is obvious. Note that if $k = 0$ then

$$G''(r) = 2Q'(r) + rQ''(r) = -\frac{3(1-\alpha)\varphi}{2(\varphi - r)^3} < 0$$

for $r < \varphi$. Thus, if $Q''(r) \geq 0$ then $G''(r) = 2Q'(r) + (r-k)Q''(r) \leq 2Q'(r) + rQ''(r) < 0$. G is therefore strictly concave on $(k, \overline{r}(\varphi, \alpha))$ which together with $G'(k) > 0$ implies that r^* is the unique solution to $G'(r) = 0$.

Suppose $k \leq \hat{k}$. Then, from Lemma 4, $G'(\underline{r}(\varphi, \alpha)^+) < 0$ and G is strictly concave on $(\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$ given $k \leq \hat{k} < \underline{r}(\varphi, \alpha)$. Since $G'(\underline{r}(\varphi, \alpha)^+) < 0$ we have $G'(r) < 0$ for all $r \in (\underline{r}(\varphi, \alpha), \overline{r}(\varphi, \alpha))$. Finally, since G is increasing on $[0, \underline{r}(\varphi, \alpha))$ we obtain $r^* = \underline{r}(\varphi, \alpha)$.

Comparative Statics. Note that $\partial \underline{r}(\varphi, \alpha)/\partial \varphi$, $\partial \underline{r}(\varphi, \alpha)/\partial \alpha$, $\partial \overline{r}(\varphi, \alpha)/\partial \varphi$, $\partial \overline{r}(\varphi, \alpha)/\partial \alpha > 0$. Further, if $k \in (\hat{k}, \overline{r}(\varphi, \alpha))$ then $G'(r^*) = 0$ and $G''(r^*) < 0$ (see the proof of Proposition 3). Implicit differentiation gives

$$\begin{aligned}
\frac{\partial r^*}{\partial k} &= -\frac{-Q'(r^*)}{G''(r^*)} > 0 \\
\frac{\partial r^*}{\partial \varphi} &= -\frac{\partial Q(r^*)/\partial \varphi + (r^* - k)\partial Q'(r^*)/\partial \varphi}{G''(r^*)} > 0 \\
\frac{\partial r^*}{\partial \alpha} &= -\frac{\partial Q(r^*)/\partial \alpha + (r^* - k)\partial Q'(r^*)/\partial \alpha}{G''(r^*)} > 0,
\end{aligned}$$

where the sign of the derivative follows since the denominator is negative and the numerator can be shown to be positive in all three cases. Hence, whenever $k < \overline{r}(\varphi, \alpha)$, r^* is strictly increasing in k, α and φ.

Finally, the following remark shows that the monopolist's profits in this setup are smaller than under full information.

Remark 8 *Let $\hat{G} := \alpha(\varphi - k)$ be the monopolist's profit under full information. Then $\hat{G} > G(r^*)$ for all $k \in [0, \varphi)$.*

Proof. (i) If $k \in [0, \hat{k}]$, then $G(r^*) = \underline{r}(\varphi, \alpha) - k$. Simple calculations show that $\alpha(\varphi - k) > \underline{r}(\varphi, \alpha) - k$. (ii) If $k \in [\overline{r}(\varphi, \alpha), \varphi)$ then $G(r^*) = 0 < \alpha(\varphi - k) = \hat{G}$. Consider $k \in (\hat{k}, \overline{r}(\varphi, \alpha))$. Note that $\frac{\partial \hat{G}}{\partial k} = -\alpha$ and $\frac{\partial G(r^*)}{\partial k} = -Q(r^*)$. We also know that $Q(r^*)$ is continuously decreasing in k. This implies that $\hat{G} - G(r^*)$ is concave in k and, by (i) and (ii), we thus have $\hat{G} > G(r^*)$ for all $k \in (\hat{k}, \overline{r}(\varphi, \alpha))$. ■

12 Appendix F: Biased comparisons

Here, we analyze the consequences of introducing biases in the comparison signal. We make the simplifying assumption that the comparison bias does not depend on the observed action or the observed individual. We do allow, however, the bias to vary across types. Let ε_τ be the bias of type τ individuals. Therefore, the expected value of the comparison signal observed by a type τ individual who samples a type τ' individual choosing c is $\pi_{\tau\tau'c} = \pi_{\tau c} - \pi_{\tau' c} + \varepsilon_\tau$, for all $\tau, \tau' \in T$ and $c \in S$. If $\varepsilon_\tau > (<)0$, then type τ individuals have positive (negative) biases. As in the benchmark model, we assume that the comparison signal takes values in a finite set with minimum -1 and maximum 1. Notice that this imposes $\varepsilon_\tau \in [\pi_{\tau' c} - \pi_{\tau c} - 1, 1 + \pi_{\tau' c} - \pi_{\tau c}]$ for all $c \in S$. This yields $L_{cd}(\tau) = \frac{1}{2} + \beta(\pi_{\tau d} - \pi_{\tau c}) + \beta\varepsilon_\tau$ and therefore $L_{ba}(A) = U + \beta\varepsilon_A$, $L_{ab}(A) = 1 - U + \beta\varepsilon_A$, $L_{ba}(B) = 1 - D + \beta\varepsilon_B$ and $L_{ab}(B) = D + \beta\varepsilon_B$. It follows that if individuals have positive biases, they switch more often, both to their optimal and suboptimal action. If they have negative biases, they are instead more reluctant to switch. The system of differential equations becomes

$$\dot{p} = \alpha p(1-p)(2U-1) + (1-\alpha)((1-p)(1-q)(U+\beta\varepsilon_A) - pq(1-U+\beta\varepsilon_A)) \tag{23}$$

$$\dot{q} = (1-\alpha)q(1-q)(2D-1) + \alpha((1-q)(1-p)(D+\beta\varepsilon_B) - qp(1-D+\beta\varepsilon_B)). \tag{24}$$

Qualitatively, system (23)-(24) behaves similarly to (4)-(5). Depending on the parameters, the system converges to either a corner or an interior rest point. Let

$$\underline{\alpha}^\varepsilon(\varepsilon_A, \varepsilon_B) := \frac{(1-U+\beta\varepsilon_A)(2D-1)}{U+D-1+\beta\varepsilon_B(2U-1)+\beta\varepsilon_A(2D-1)}$$

and

$$\overline{\alpha}^\varepsilon(\varepsilon_A, \varepsilon_B) := \frac{(U+\beta\varepsilon_A)(2D-1)}{U+D-1+\beta\varepsilon_B(2U-1)+\beta\varepsilon_A(2D-1)}.$$

Also let $\hat{p}(\varepsilon_A, \varepsilon_B) := \frac{(\beta\varepsilon_A+U)(\alpha-\underline{\alpha}^\varepsilon(\varepsilon_A,\varepsilon_B))}{\alpha(2U-1)}$ and $\hat{q}(\varepsilon_A, \varepsilon_B) := \frac{(\beta\varepsilon_B+D)(\overline{\alpha}^\varepsilon(\varepsilon_A,\varepsilon_B)-\alpha)}{(1-\alpha)(2D-1)}$. Using an argument analogous to that in Section 3 we obtain the following result.

Remark 9 *For all paths such that $(p(0), q(0)) \notin \{(0,1), (1,0)\}$,*

$$\lim_{t\to\infty}(p(t), q(t)) = \begin{cases} (0,1) & \text{if} \quad \alpha < \underline{\alpha}^\varepsilon(\varepsilon_A, \varepsilon_B) \\ (\hat{p}(\varepsilon_A, \varepsilon_B), \hat{q}(\varepsilon_A, \varepsilon_B)) & \text{if} \quad \underline{\alpha}^\varepsilon(\varepsilon_A, \varepsilon_B) < \alpha < \overline{\alpha}^\varepsilon(\varepsilon_A, \varepsilon_B) \\ (1,0) & \text{if} \quad \alpha > \overline{\alpha}^\varepsilon(\varepsilon_A, \varepsilon_B). \end{cases}$$

We now consider how the predictions of the model respond to changes in ε_A and ε_B. For concreteness we focus on ε_A. First, $\underline{\alpha}_1^{\varepsilon}(\varepsilon_A, \varepsilon_B), \overline{\alpha}_1^{\varepsilon}(\varepsilon_A, \varepsilon_B) > 0$. This means that, as type A individuals' bias increases, the system converges to $(0,1)$ for a larger set of parameter configurations and to $(1,0)$ for a smaller set of parameters. Next, we obtain that

$$\hat{p}_1(\varepsilon_A, \varepsilon_B) = \frac{\varepsilon_A(\alpha - \underline{\alpha}^{\varepsilon}(\varepsilon_A, \varepsilon_B)) - (\varepsilon_A + U)\underline{\alpha}_1^{\varepsilon}(\varepsilon_A, \varepsilon_B)}{\alpha(2U-1)} < 0.$$

Hence, the outcome becomes worse for type A individuals as their bias increases. On the other hand $\hat{q}_1(\varepsilon_A, \varepsilon_B) > 0$, i.e., type B individuals benefit from type A individuals' bias. If we take as a starting point $\varepsilon_A = \varepsilon_B = 0$, these results imply that type A individuals are affected negatively by their bias, while type B individuals benefit from this. On the other hand, type A individuals obtain a better outcome when their bias decrease, while type B individuals are negatively affected by this.

There is a simple intuition behind these results. As we see in equations (23)-(24), the effect of the bias on $\dot{p}$ depends on p and q. If the biases are positive, there is a positive effect for small values of p and q (more precisely, below the line $q = 1 - p$) and a negative effect for large values. The isoclines in the phase diagrams are always above the line $q = 1 - p$, however. Intuitively, when q is large, it is difficult to find a type B individual choosing a. At the same time, when p is large, there is only a small fraction of type A individuals choosing b and who are therefore candidates to switch to action a. Hence, switching towards action a in A is small, and the positive effect of the bias on this switching is, therefore, relatively unimportant. In contrast, there are many type A individuals choosing a that observe type B individuals choosing b, and hence the effect of the bias on these switches is larger. The result is that a positive bias causes a net increase in switching towards b among type A individuals, which in the end makes A worse off. The opposite occurs when type A individuals have negative biases. A negative bias reduces switching, and at large values of p and q this inhibits switching away from a to a greater extent than it inhibits switching away from b. This leads to a larger fraction of type A individuals choosing their optimal action in the long run. By the same logic as in the benchmark model, whatever causes a stronger motion towards a among type A individuals has a negative effect on type B individuals. Hence, when type A individuals have negative biases and obtain a better outcome, type B individuals are worse off. Analogously, when type A individuals have positive biases and obtain a worse outcome, this benefits type B individuals.

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