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Jürgen Eichberger, Simon Grant,
and Jean-Philippe Lefort

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Abstract

This paper shows that, for CEU preferences, the axioms *consquentialism, state independence* and *conditional certainty equivalent consistency* under updating characterise a family of capacities, called *Genralised Neo-Additive Capacities (GNAC).* This family contains as special cases among others neo-additive capacities as introduced by Chateauneuf, Eichberger, and Grant (2007), Hurwicz capacities, and ε-contaminations.

Moreover, we will show that the convex version of a GNAC is the only capacity for which the core of the Full-Bayesian Updates of a capacity, introduced by Jaffray (1992), equals the set of Bayesian updates of the probability distributions in the core of the original capacity

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Adress for Correspondence: Department of Economics, University of Heidelberg, Germany

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1 Introduction

A major problem when modelling ambiguity of a decision maker in a dynamic context lies in the well-known precarious relationship between updating capacities or multiple priors, dynamic consistency and consequentialism. Early work by Epstein and LeBreton (1993) and Eichberger and Kelsey (1996) showed that updating Choquet Expected Utility (CEU) preferences, which satisfy consequentialism, in a dynamically consistent way implied additive beliefs. Even if dynamic consistency was constrained to an event tree, ambiguous beliefs modelled by a capacity were possible only on the final partition of events (Sarin and Wakker (1998) and Eichberger, Grant, and Kelsey (2005)). In the context of ambiguity models with multiple priors, Epstein and Schneider (2003) found that the set of priors had to fulfill a fairly restrictive rectangularity condition in order to guarantee dynamically consistent preferences. In particular, the original Ellsberg paradox cannot be explained with rectangular sets of priors.

In the light of these results, there are essentially two ways to proceed. Either one can abandon consequentialism, and all the models relying on it like CEU and multiple priors, or to give up dynamic consistency. In this paper, we retain consequentialism. The former route has been explored by Hanany and Klibanoff (2007).

In the spirit of Gilboa and Schmeidler (1993), we consider a preference relation and the family of its updated preferences which satisfy the two axioms *Consequentialism* and *State Independence.* For the case where beliefs can be described by multiple priors, as in the max-min expected utility (MMEU) preference model of Gilboa and Schmeidler (1989), Pires (2002) proved that these two axioms plus a third axiom that Eichberger, Grant, and Kelsey (2007) refer to as *Conditional Certainty Equivalent Consistency (CCEC)* are equivalent to the *Full Bayesian Updating* of all prior probabilities. If the preference relation can be represented by a Choquet integral and beliefs by a capacity, as in the Choquet Expected Utility (CEU) preference model of Schmeidler (1989), Eichberger, Grant, and Kelsey (2007) and Horie (2007) established that *Consequentialism*, *State Independence,* and a weakening of CCEC that Horie refers to as *Conditional Certainty Equivalent Consistency for Binary Acts*, are equivalent to *Full Bayesian Updating* of the capacity as suggested by Jaffray (1992) and Walley (1991).[1]

In this paper, we characterize the family of capacities for which the three initial axioms of Pires (2002) hold, that is, where *CCEC* is not restricted to binary acts. We find a class of capacities which is slightly more general than the family of *neo-additive capacities* which were introduced and axiomatized in Chateauneuf, Eichberger, and Grant (2007). For CEU preferences for which the associated capacity is a neo-additive capacity, the Choquet expected utility of an act can be expressed as a convex combination of the expected utility with respect to an additive probability distribution and the *Hurwicz criterion* (Hurwicz (1951)) which itself is a convex combination of the utility values of the best and the worst outcomes.

The Choquet expected utility of an act with respect to a *generalized neo-additive capacities (GNAC)* combines the subjective expected utility of the act and the utility values of the best and the worst outcomes linearly, but the combinations need no longer to be convex but merely monotone. Moreover, we can show that *convex* GNACs are the only capacities for which the core of the Full Bayesian update of a capacity coincides with the set of Bayesian updates of the probabilities in the core of the original capacity. These results provides a further justification for *neo-additive capacities* as a useful restriction on the Choquet expected utility approach.[2]

The paper is organised as follows. After introducing in the next section the basic framework, we prove in section 3 that CEU preferences satisfy Axiom CCEC if and only if the capacity v is a GNAC. Section 4 discusses several special cases of GNACs before concluding the paper with some open questions. Unless otherwise stated the proofs are given in the appendix.

2 The Framework

Let S be a finite set of states of the world, $\Sigma = \mathcal{P}(S)$, the set of events in S. For any $E \in \Sigma$, let

[1] Horie (2007) showed that the necessary conditions in Eichberger, Grant, and Kelsey (2005) were too stong and suggested the appropriate weakening of the CCEC where CCEC is restricted to hold only for binary acts.

[2] In a recent paper, Chateauneuf, Faro, Gajdos, and Jaffray (2009) investigate robust updating rules. From the perspective of robust statistics introduced by Huber (1981), they show that updated (ε, δ)-*contaminations* are the only class of capacities whose core coincides with the set of Bayesian updates of the probabilities in the core of the original capacity. (ε, δ)-*contaminations* are a special case of generalised neo-additive capacities. In this regard, proposition 2 of Chateauneuf, Faro, Gajdos, and Jaffray (2009) overlaps with Proposition 4.3 in this paper.

E^c denote the complement of E. Let X be a set of outcomes. An *act* is a function $f : S \to X$, and $\mathcal{F}$ denotes the set of such acts. Given any two acts f and g in $\mathcal{F}$ and any event E in Σ, we denote by $f_E g \in \mathcal{F}$ the act defined as $f_E g(s) = f(s)$ if $s \in E$ and $f_E g(s) = g(s)$ otherwise. For notational convenience, we will not distinguish between the outcome $x \in X$ and the constant act $x \in \mathcal{F}$, defined as $x(s) = x$ for all $s \in S$.

Binary acts are an important special case. For any two outcomes x and y in X and any event E in Σ, the binary act $x_E y$ is defined as

$$x_E y(s) = \begin{cases} x & \text{if} \quad s \in E \\ y & \text{otherwise} \end{cases} .$$

A *capacity* v is a set function from Σ to $\mathbb{R}$ with $v(\emptyset) = 0$, $v(S) = 1$ and $v(A) \leq v(B)$ for all $A \subset B$, A and B in Σ.

Given a *von Neumann-Morgenstern utility function* $u : X \to \mathbb{R}$, two acts f and g are *comonotonic* with respect to u, if for all pairs of states s and s' in S,

$$[u(f(s)) - u(f(s'))][u(g(s)) - u(g(s'))] \geq 0.$$

The *Choquet Expected Utility (CEU)* of an act f with respect to the capacity v is given by

$$CEU(f, v) = \int_{-\infty}^{0} (v(u(f(s)) \geq t) - 1)dt + \int_{0}^{+\infty} v(u(f(s)) \geq t)dt.$$

Since acts are finite-valued they can be written as $f = \sum_{i=1}^{n} x_i A_i$, where, in a convenient abuse of notation, we denote by A_i both the set $A_i \in \Sigma$ and the indicator function of the set A_i. That is, $A_i(s) = 1$ for $s \in A_i$ and 0 otherwise. For any $f \in \mathcal{F}$, we denote by $[f]$ the subset of acts comonotonic to f, which are measurable with respect to the partition $A_1, ...A_n$. Without loss of generality, suppose that the finite outcomes $x_i \in f(S)$ are ordered such that $u(x_i) \leq u(x_{i+1})$, then

$$\begin{aligned} CEU(f, v) &= \sum_{i=1}^{n} u(x_i) \cdot [v(A_i \cup A_{i+1} \cup ... \cup A_n) - v(A_{i+1} \cup A_{i+2} \cup ... \cup A_n)] \\ &= \sum_{i=1}^{n} u(x_i) \cdot m_{[f]}(A_i), \end{aligned}$$

with $m_{[f]}(A_i) := [v(A_i \cup A_{i+1}..A_n) - v(A_{i+1} \cup A_{i+2}..A_n)]$. Note that $\sum_{i=1}^{n} m_{[f]}(A_i) = 1$ holds. Thus, one can view the Choquet integral as determined by a set of probability distributions m, one for each possible ordering of outcomes.

Throughout this paper, we will consider preference relations $\succsim$ on $\mathcal{F}$ which can be represented by a *Choquet Expected Utility (CEU)* functional,

$$f \succsim g \quad \text{if and only if} \quad CEU(f, v) \geq CEU(g, v).$$

An event $E \in \mathcal{E}$ is *null* (*universal*) if $x_E y \sim y$ ($x_E y \sim x$) for all pairs of outcomes $x, y \in X$ with $x \succ y$. An event E is *essential* if for some $x, y \in X$, $x \succ x_E y \succ y$. Let $\mathcal{N}$, $\mathcal{U}$ and $\mathcal{E}^*$ denote the sets of null, universal and essential events, respectively.[3] The following axiom, which is not implied by CEU, will be assumed to hold throughout the paper.

Axiom 0 (*Null-Event Consistency*)

For all pairs of outcomes x and y such that $x \succ y$, $x_E y \sim y$ implies $y_E x \sim x$.

2.1 Updating preferences

Consider a family of preference relations $\succsim_E$ on $\mathcal{F}$ which represent the decision maker's preferences after it becomes known that a non-null event E has occurred. The ex-ante unconditional preference relation on $\mathcal{F}$ will be denoted by $\succsim$.

For preferences which are additive, that is, represented by a CEU functional with an additive capacity v, standard Bayesian updating satisfies the following three axioms.

Axiom SI *State Independence*

For any two outcomes x, $y \in X$, and any non-null event $E \notin \mathcal{N}$,

$$x \succsim y \Leftrightarrow x \succsim_E y.$$

Axiom C *Consequentialism*

For any two acts f, $g \in \mathcal{F}$, and any event $E \in \Sigma$,

$$\text{if } f = g \text{ on } E, \quad \text{then} \quad f \sim_E g.$$

Axiom DC *Dynamic Consistency*

[3] The definition of null, universal and essential events follows Ghirardato and Marinacci (2002).

For any acts $f, g \in \mathcal{F}$ and any event $E \in \Sigma$,

$$f \succ_E g \quad \text{if and only if} \quad f_E g \succ g.$$

State independence requires the conditional preferences over outcomes to agree with the unconditional preferences over outcomes. Consequentialism rules out effects on future choices from outcomes which would have become relevant in the event E^c which did not happen. Dynamic consistency links conditional and unconditional preferences by requiring that preferences after E occurred remain consistent with ex-ante preferences.

It is well known (Ghirardato (2002)), that together the three axioms imply for a CEU decision maker that the utility function remains unchanged for the conditional preferences and the capacity v is additive and updated by Bayes rule. As we wish to retain the property that the ordinal ranking over outcomes is not affected by which event we condition upon, to have an updating rule that leaves room for uncertainty represented by a non-additive capacity we must relax either consequentialism or dynamic consistency.

Retaining consequentialism, Pires (2002) proposes a weaker version of DC, *conditional certainty equivalent consistency* which restricts the act g of the classical DC axiom to be constant.

Axiom CCEC *Conditional Certainty Equivalent Consistency*

For any event $E \neq \varnothing$, any outcome $x \in X$, and any act f in $\mathcal{F}$,

$$f \sim_E x \quad \text{if and only if} \quad f_E x \sim x. \tag{1}$$

If the acts f in Equation 1 are restricted to binary acts, then we refer to Axiom CCEC as *Conditional Certainty Equivalent Consistency for Binary Acts.*

Applying Axiom 0 to a binary act, reveals immediately the implication that the complement of a null event is universal. Although this appears to be a natural assumption if capacities are supposed to represent beliefs, together with Axiom CCEC (see Lemma 2.1 below) it implies that the union of two null events must be also null. In the context of ambiguity, this conclusion appears less innoxious. After all, ambiguity may manifest itself in a decision maker's inability to assign probability values to subevents, even if s/he feels capable of such a judgment for the union of the events. This implication, however, is driven by Axiom CCEC and, hence, has to be judged in context with the latter.

The following lemma[4] shows that the updating axiom, Axiom CCEC, in combination with Axiom 0 yields a set of null events which is closed under the union operation.

Lemma 2.1 *If Axiom 0 and Axiom CCEC for to binary acts hold, then* $v(A) = v(B) = 0$ *implies* $v(A \cup B) = 0$.

Proof. Let A and B be such that $v(A) = v(B) = 0$ and $v(A \cup B) \neq 0$ holds. By Axiom 0, $v(A^c) = 1$. For $x, y, z \in X$ with $x \prec y \prec z$, let $f = xA + zB$ and, for $E = A \cup B$, let $f_E y = xA + yE^c + zB$. Then $\int f_E y dv = x(1 - v(A^c)) + y(v(A^c) - v(B)) + zv(B) = y$. Hence, $f_E y \sim y$. The same reasoning holds for any $y' \in X$ with $y' \neq y,\ x \prec y' \prec z$. Hence, $f_E y' \sim y'$. By Axiom CCEC, $f \sim_E y \sim_E y'$ for all $y' \in X,\ x \prec y' \prec z$. W.l.o.g. assume $x \prec y \prec y' \prec z$. By Axiom CCEC, $y' \sim_E y$ if and only if $y'_E y \sim y$. Hence, $\int \left(y'_E y\right) dv = y(1 - v(E)) + y'v(E)) = y$ and $v(E) = v(A \cup B) \neq 0$ imply $y \sim y'$, a contradiction. ■

3 Generalized Neo-Additive Capacities (GNACs)

For multiple-prior preferences, Pires (2002) proved that *state independence, consequentialism* and *conditional certainty equivalent consistency* imply the Full Bayesian updating rule, where each probability distribution in the set of priors is updated according to Bayes rule. In the CEU context, straightforward application of the definition reveals that CEU preferences satisfy Axiom SI and Axiom C. Moreover, from Chateauneuf, Eichberger, and Grant (2007) and Horie (2007) we know that *Axiom CCEC restricted to binary acts* implies that the capacities of CEU preferences are updated according to the *Full Bayesian Updating rule* suggested by Jaffray (1992) and Walley (1991).

We will show now that Axioms 0 and Axiom CCEC in its full strength determine a class of capacities similar to neo-additive capacities, as axiomatized in Chateauneuf, Eichberger, and

[4] Note that the proof uses only binary act consistency, the weaker notion of Axiom CCEC which was suggested by Horie (2007). Notice that Axiom 0 would also have to be assumed for the main result in Horie (2007), if it were extended to null sets other than the empty set.

Grant (2007). We will call these capacities *Generalized Neo-additive Capacities,* or *GNAC*s for short.

Null events.

We do not restrict capacities to have non-zero values for events other than the empty set. This generality requires us to introduce some regularity conditions.

Definition 3.1 *A collection of sets $\mathcal{N} \subset \Sigma$ is said to exhibit the* null-property *if (i) $\varnothing \in \mathcal{N}$ and $S \notin \mathcal{N}$; (ii) $A \in \mathcal{N} \Rightarrow B \in \mathcal{N}$, for all $B \subset A$; and (iii) $A \in \mathcal{N}$ and $B \in \mathcal{N} \Rightarrow A \cup B \in \mathcal{N}$.*

The collection of sets $\mathcal{N}$ exhibits the null-property if it contains the empty set but not the state-space, if all subsets of a null set are also in $\mathcal{N}$, and if it is closed under union.

A capacity is congruent with a collection of null events $\mathcal{N}$ if it assigns a capacity value of zero to every element in this collection[5]. Equivalently, if an event has positive capacity value then it cannot be an element of $\mathcal{N}$.

Definition 3.2 *Fix a collection of sets $\mathcal{N}$ that exhibits the null-property. The capacity μ is* congruent with respect to $\mathcal{N}$, *if $A \in \mathcal{N} \Rightarrow \mu(A) = 0$ (or equivalently, $\mu(A) > 0 \Rightarrow A \notin \mathcal{N}$).*

Exactly congruent is the stronger requirement that an event has zero capacity if and only if it is in $\mathcal{N}$. The capacity of complete ambiguity defined by $\mu(E) = 0$ for all $E \neq S$ provides a simple example of a capacity which is congruent, but not exactly congruent with $\mathcal{N} := \{\varnothing\}$.

Generalised Neo-Additive Capacities (GNAC).

A *Generalized Neo-Additive Capacity (GNAC)* can now be defined as a linear transformation of an additive probability distribution which satisfies the monotonicity condition.

Definition 3.3 *Let $\mathcal{N}$ be a collection of null events with the null-property, π a finitely additive probability measure on (S, Σ), and a and $b > 0$ a pair of numbers satisfying* $\min_{E \notin \mathcal{N}} [a + b\pi(E)] \geq$

[5] Capacities congruent with a collection of null events satisfying the null-property are known as null additive set functions, see Pap (1995).

0 *and* $\max_{E\notin\mathcal{N}}\left[a+b\left(1-\pi\left(E\right)\right)\right]\leq 1$, *then a* Generalized Neo-Additive Capacity $v(\cdot|\mathcal{N},\pi,a,b)$ *is defined as,*

$$v(E|\mathcal{N},\pi,a,b):=\begin{cases} 0 & \textit{if } E\in\mathcal{N} \\ a+b\pi\left(E\right) & \textit{if } E\notin\mathcal{N} \textit{ and } E^{c}\notin\mathcal{N} \\ 1 & \textit{if } E^{c}\in\mathcal{N} \end{cases}.$$

The Choquet expected value of an act f with respect to the GNAC $v(E|\mathcal{N},\pi,a,b)$ is easily computed as

$$\begin{aligned} CEU(f,v)= \quad & b \int_{\{s,\exists t\in S, f(s)\leq f(t), v(\{t\})\neq 0\}} (u\circ f)d\pi \\ & +a\cdot\max\{x|\ x\in\left(u\circ f\right)\left(\{s,\nu(\{s\})\neq 0\}\right)\} \\ & +(1-a-b)\cdot\min\{x|\ x\in\left(u\circ f\right)\left(\{s,\nu(\{s\})\neq 0\}\right)\}). \end{aligned}$$

In Chateauneuf, Eichberger, and Grant (2007) the following property of a capacity was introduced.

Property A $\quad v(E\cup F)-v(F)=v(E\cup G)-v(G)$ is satisfied for all events E, F and G such that $v(F)\neq 0$, $v(F\cup E)\neq 1$, $v(G)\neq 0$ and $v(G\cup E)\neq 1$.

It is an easy exercise to check that a GNAC satisfies Property A.

We now state our main result.

Proposition 3.1 *For a CEU preference relation represented by a capacity ν which satisfies Axiom 0, Axiom SI, Axiom C, the following statements are equivalent:*

(i) Axiom CCEC is satisfied.

(ii) v is updated with FBU and Property A is satisfied.

(iii) v is updated with FBU and v is a GNAC.

The following remark shows that small generalizations of the result in Proposition 3.1 are possible.

Remark 3.1 *(i) It is worth noting that our proof uses only one way of Axiom CCEC, namely $f\sim_{E}x\Rightarrow f_{E}x\sim x$.*

(ii) In the statement of Axiom CCEC, we could replace the constant act x by a slightly more

general act:

Alternative Axiom*: Suppose* $\varnothing \neq \arg\min_{s\in S} f(s) \cap \arg\min_{s\in S} g(s)x \subset A$ *and* $\varnothing \neq \arg\max_{s\in S} f(s) \cap \arg\max_{s\in S} g(s) \subset A$*, then for any* $h \in \mathcal{F}$ *such that*

$$\max\left\{\min_{s\in S} f(s), \min_{s\in S} g(s)\right\} \leq \min_{s\in S} h(s), \quad \max_{s\in S^c} h(s) \leq \min\left\{\max_{s\in S} f(s), \max_{s\in S} g(s)\right\}.$$

$$f \sim_A g \quad \text{if and only if} \quad f_A h \sim g_A h.$$

This alternative axiom is stronger than CCEC but, for CEU preferences, it is equivalent to CCEC. Hence, for CEU preferences, Axiom CCEC implies GNAC which in turn implies the alternative axiom. ■

Remark 3.2 *Both neo-additive capacities and GNACs are based on an additive probability distribution, but for a GNAC this probability distribution need no longer be congruent with the null set* $\mathcal{N}$*. For a GNAC it is possible that exists an event* E *such that* $\pi(E) \neq 0$, $v(E) = 0$ *and that for any essential event* F, $v(F \cup E) = v(F) + b\pi(E) \neq v(F)$*. The existence of such event has the following implication for the computation of the Choquet integral: consider an act* f*, then there are two possibilities:*

- either there is at least one essential event on which the outcome is not inferior to the outcome on E*, in which case the capacity value of* E *in the Choquet integral of* f *will be* $v(E \cup F) - v(F) = v(F) + b\pi(E)$*,*

-or the best outcome of f *lies only on* E*, then the capacity value of* E *in the Choquet integral of* f *is* $v(E) = 0$*.*

GNACs can describe a situation where the decision maker ignores events with the highest outcomes. Ignoring events with the best outcome can be viewed as a form of pessimism.

The Choquet expected value of an act f *with respect to the GNAC* $v(E|\mathcal{N}, \pi, a, b)$ *is computed as*

$$\begin{aligned} CEU(f, v) = \quad & b \int_{\{s, \exists t \in S, f(s) \leq f(t), v(\{t\}) \neq 0\}} (u \circ f) d\pi \\ & + a \cdot \max\{x | \ x \in (u \circ f)(\{s, \nu(\{s\}) \neq 0\})\} \\ & + (1 - a - b) \cdot \min\{x | \ x \in (u \circ f)(\{s, \nu(\{s\}) \neq 0\})\}). \end{aligned}$$

In order to impose this congruence, we can use the following strengthening of axiom 0:

Axiom 0' *(*Strong Null-Event Consistency*)*
For any three outcomes $x, y, z \in X$ with $x \succ y \succ z$, any null event $N \in \mathcal{N}$ and any event $E \in \Sigma$, s.t. $E \cap N = \emptyset$,

$$y_N\left(x_E z\right) \sim x_E z. \tag{2}$$

Null-event consistency requires a null event not to affect the evaluation of a bet as long as it carries a non-extreme outcome[6]*. For $E = \varnothing$, Equation 2 yields the definition of a null event. The following lemma shows that Axiom 0 guarantees that the capacity of the CEU is a null-additive set functions, see Pap (1995).*

Lemma 3.2 *CEU preferences satisfy Axiom 0' if and only if for any null event $N \in \mathcal{N}$ and any event $E \in \Sigma$, s.t. $E \cap N = \emptyset$, $v(E \cup N) = v(E)$.*
Proof. *Applying CEU to Equation 2 one has*

$$\begin{aligned} & u(x)v(E) + u(y)[v(E \cup N) - v(E)] + u(z)[1 - v(E \cup N)] \\ = \quad & u(x)v(E) + u(z)[1 - v(E)] \\ \Longleftrightarrow \quad & \quad [u(y) - u(z)][v(E \cup N) - v(E)] = 0. \end{aligned}$$

■

With this strong null event consistency axiom, property D of the appendix is fulfilled. Therefore lemma A.1 can be applied and the probability π is also congruent with $\mathcal{N}$. The Choquet integral of such GNACs combines then the expectation computed according to the probability with max and min:

$$\begin{aligned} CEU(f,v) \quad = \quad & b \int_S (u \circ f) d\pi + a \cdot \max\{x |\, x \in (u \circ f)\left(\{s, \nu(\{s\}) \neq 0\}\right)\} \\ +(1-a-b) \cdot \min\{x |\, x \quad \in \quad & (u \circ f)\left(\{s, \nu(\{s\}) \neq 0\}\right)\}). \end{aligned}$$

[6] The property imposed by Axiom 0 is implied by Axiom 5 in Chateauneuf, Eichberger, and Grant (2007)

4 GNACs and updating

Assuming CEU preferences which satisfy null-event consistency and consequentialism, updating rules impose conditions on unconditional preferences which may characterise these preferences. Thus, dynamic consistency implies additive capacities. Similarly, as the previous section has shown, CCEC implies that capacities have the form of Generalised Neo-Additive Capacities. GNACs are linear transformations of an additive probability distribution. Several well-known examples of capacities can be obtained as special cases by putting additional constraints on the parameters a and b.

1.	Subjective Expected Utility (Savage (1954)):	$a = 0,$	$b = 1.$
2.	Simple capacities, ε-Contaminations:	$a = 0,$	$b \leq 1.$
3.	(ε, δ)-Contaminations		
	(Chateauneuf, Faro, Gajdos, and Jaffray (2009)):	$-1 \leq a \leq 0,$	$b < 1.$
4.	Hurwicz capacity (Hurwicz (1951)):	$0 \leq a \leq 1,$	$b = 0.$
4.	Convex capacity:	$a \leq 0,$	$a + b \leq 1.$
5.	Concave capacity:	$a \geq 0,$	$a + b \geq 1.$
6.	Cavex capacity (Wakker (2001), p. 1049):	$a \geq 0,$	$a + b \leq 1.$

The updating axiom CCEC, however, also has implications for the updated preferences. In this section, we will derive and discuss some properties for the updates of GNAC. Eichberger, Grant, and Kelsey (2007) and Horie (2007) show that Axiom CCEC resstricted to binary acts implies that the capacities of CEU preferences must be updated according to the *Full-Bayesian Updating rule (FBU)* of Jaffray (1992). Given the information that E has occurred, the FBU-rule is defined as the capacity ν_E,

$$\begin{aligned} v_E(A) \quad &: = \frac{v(A \cap E)}{v(A \cap E) + 1 - v(A \cup E^c)} \\ &= \frac{v(A \cap E)}{v(A \cap E) + \overline{v}(A^c \cap E)}, \end{aligned} \tag{3}$$

where $\overline{v}(A) := 1 - v(A^c)$ denotes the dual capacity of v.

Properties of the FBUs depend on the classification according to $a \gtreqless 0$ and $a + b \gtreqless 1$.

	$a+b\leq 1$	$a+b\geq 1$
$a\leq 0$	convex	'vexcave'
$a\geq 0$	cavex	concave

4.1 Convex GNACs $(a \leq 0, a+b \leq 1)$

Let v be a convex capacity, and let $C(v) = \{p \in \Delta(S)|\ p \geq v\}$ denote its core. It is well-known (Schmeidler (1989)) that $CEU(f,v) = \min_{p\in C(v)} \int (u \circ f)\, dp$ in this case. In the light of the result by Pires (2002) that an axiom like CCEC applied to a multiple-priors model yields that the set of priors after updating equals the set of the Bayesian updates of the priors, one may be inclined to think that a similar result would hold for CEU preferences. As Horie (2007) points out, however, and as one easily checks, for each $p \in C(v)$ and any event E such that $p(E) > 0$, we have $C(v)_E \subseteq C(v_E)$, where $C(v)_E$ denote the set of Bayesian updates with respect to E of the probabilities in the core $C(v)$. To see this notice that

$$\frac{p(A)}{p(E)} - v_E(A) = \frac{p(A)}{p(E)} - \frac{v(A)}{v(A)+\overline{v}(A^c \cap E)} = \frac{p(A)\overline{v}(E\backslash A) - v(A)p(E\backslash A)}{p(E)(v(A)+\overline{v}(E\backslash A))}.$$

As $p \in C(v)$, we have $p(A) \geq v(A)$ and $p(E\backslash A) \leq \overline{v}(E\backslash A)$. Hence, $p(A)/p(E) \geq v_E(A)$. Horie (2007) also shows by example that there exists a convex capacity v' for which $C(v')_E \neq C(v'_E)$. Thus, in general, $C(v)_E \neq C(v_E)$.

The reverse inclusion $C(v_E) \subseteq C(v)_E$ holds, however, if the capacity is a *convex GNAC* and if there are at least *three states*.

Proposition 4.1 *If $|S| > 3$, then $C(v)_E = C(v_E)$ if and only if v is a* convex *GNAC.*

As an immediate consequence of Proposition 4.1 is the representation of the conditional preferences by the same type of functional. Since the FBU v_E of a convex capacity v is also convex, it follows for a convex GNAC that

$$CEU(f, v_E) = \min_{p\in C(v)_E} \int (u \circ f)\, dp.$$

A couple of remarks may help to put Proposition 4.1 into the context of related work.

Remark 4.1 *The following remarks are in order:*

(i) Studying robust updating rules, Chateauneuf, Faro, Gajdos, and Jaffray (2009) derive the result of Proposition 4.1 for (δ, ε)-Contaminations.

(ii) For a finite state space S, there exist convex GNACs which are not ε-contaminations. For example, $|S| = 4$ and $\pi(E) = \frac{|E|}{|S|}$, then $v = \frac{6}{5}\pi - \frac{1}{5}$ is convex, but not an $\varepsilon-$contamination. With a non-atomic state space S, however, monotonicity implies that the only convex GNAC are ε-contaminations.

(iii) Proposition 4.1 provides necessary and sufficient conditions for capacities to guarantee $C(v_E) = C(v)_E$. An alternative condition can be found in Theorem 2 of Jaffray (1992). Proposition 4.1, however, holds for convex capacities whereas Jaffray's Theorem 2 is true only for belief functions.

(iv) If $|S| = 3$ holds, then $C(v_E) = C(v)_E$ is true for every convex capacity.

(v) If $|S| > 3$, it follows from Proposition 3.1 that the only case in which $C(v_E) = C(v)_E$ holds for convex capacities is when Axiom CCEC is true.

Proposition 4.1 shows that GNACs are the only class of capacities for which the updated probabilities of the core coincide with the core of the FBUs of the capacity. This result can be generalised to arbitrary updating rules of convex capacities for which the core of the updated capacity coincides with the Bayesian updates of the probabilities in the core of the capacity.

Proposition 4.2 *Let P_E be the set of updates on event E of a set of priors P. If P_E is the core of a convex capacity v_E then v is a GNAC.*

Proposition 4.2 shows the strength of the consistency requirements for updating rules. Nevertheless, the class of Generalised Neo-Additive Capacities together with the Full Bayesian Updating rule allow for a class of CEU representations with a reasonable degree of consistency in updating which covers a broad range of models used in economic applications.

For $a \geq 0$, convex GNACs coincide with the case of simple capacities or ε-contaminations which has been used extensively in the literature[7]. Following Schmeidler (1989), this case,

[7] Examples of applications include Dow and Werlang (1992), Eichberger and Kelsey (2002), Eichberger

has been associated with ambiguity aversion or pessimism. For GNACs, however, the convex case does not require $a \geq 0$. Monotonicity of the capacity imposes only the general constraint $\min_{E \notin \mathcal{N}} [a + b\pi(E)] \geq 0$. This constraint implies a lower bound for a, $a \geq -b \min_{E \notin \mathcal{N}} \pi(E)$ which is only equal to 0 if the probability distribution π puts probability 0 on a non-null event. The degree to which a can become negative depends, however, on the probability distribution π. The intuitive interpretation of this case not clear.

Remark 4.2 *In the convex case, the probability π has to be congruent with $\mathcal{N}$. Let $v(E) = 0$ then by convexity of v, $v(E \cup F) + v(E^c) \leq v(S) + v(F)$. As $v(E^c) = 1$ then $v(E \cup F) = v(F)$. Therefore for a convex GNAC, property D of lemma A.1 is implied by convexity.*

4.2 Concave GNAC $(a \geq 0, a + b \geq 1)$

Concave GNACs are the duals of convex GNACs. Hence, it is not surprising that the results of the previous subsection hold mutatis mutandis. For a concave capacity v and denote by $\bar{C}(v) = \{p \in \Delta(S) | \ p \leq v\}$ the *anti-core*. In this case, $CEU(f, v) = \max_{p \in \bar{C}(v)} \int (u \circ f) \, dp$. Given an event E with $p(E) > 0$, let $\bar{C}(v)_E$ be the set of Bayesian updates of the probabilities in the anti-core $\bar{C}(v)$. By analogous reasoning as in the previous case of convex capacities, we can show $\bar{C}(v)_E \subseteq \bar{C}(v_E)$. The conjugate of the capacity in Example 1 of Horie (2007) illustrates that, in general, $\bar{C}(v)_E \neq C(v_E)$. Our next proposition, which we state without formal proof, shows that $\bar{C}(v)_E = \bar{C}(v_E)$ if and only if v is a concave GNAC.

Proposition 4.3 *If $|S| > 3$, then $\bar{C}(v)_E = \bar{C}(v_E)$ if and only if v is a* concave *GNAC.*

Hence, for a concave GNAC, we have as well the representation

$$CEU(f, v_E) = \max_{p \in \bar{C}(v)_E} \int (u \circ f) \, dp.$$

4.3 Cavex GNACs $(a \geq 0, a + b \leq 1)$: Neo-additive capacities.

Wakker (2001) introduced the notion of a *cavex capacity*: "Concavity is imposed on the unlikely

and Kelsey (2004), Teitelbaum (2007), Jeleva and Rossignol (2008), and Eichberger, Kelsey, and Schipper (2009).

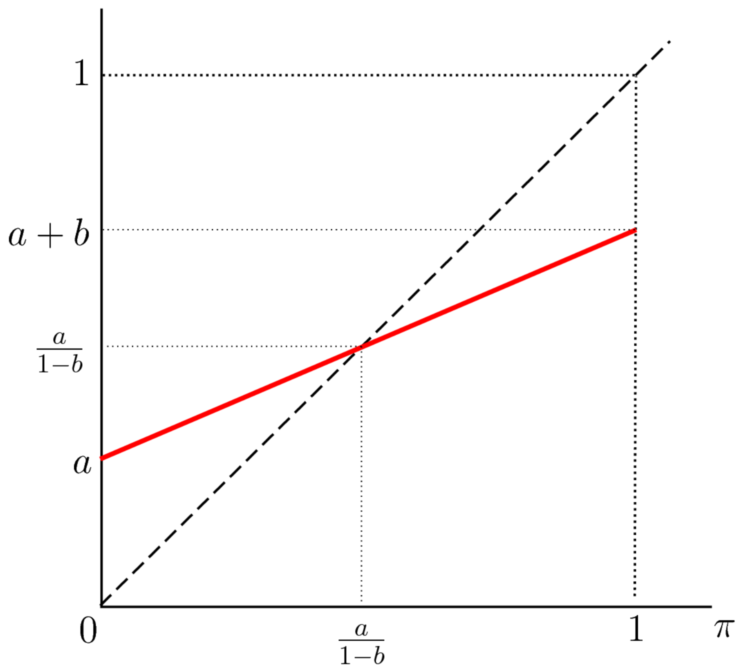

Figure 1.

events and convexity on the likely events. Henceforth, such capacities are called *cavex*." (p. 1049). Figure 1 shows the capacity value as a function of its additive part π.

A cavex GNAC is a *neo-additive capacity* as introduced and axiomatised by Chateauneuf, Eichberger, and Grant (2007). A neo-additive capacity congruent with the null sets in $\mathcal{N}$, the probability π, and the parameters $\alpha, \delta \in [0,1]$ is given by

$$v\left(E|\mathcal{N},\pi,\delta,\alpha\right)=\begin{cases} 0 & \text{if } E\in\mathcal{N} \\ \left(1-\delta\right)\pi\left(E\right)+\delta\alpha & \text{if } E\notin\mathcal{N} \text{ and } E^{c}\notin\mathcal{N} \\ 1 & \text{if } E^{c}\in\mathcal{N} \end{cases}$$

This corresponds to a cavex GNAC with $a:=\delta\alpha\geq 0$ and $b:=(1-\delta)\leq 1$.

Cavex GNACs can also be viewed as a special case of a type of capacity introduced by Jaffray and Philippe (1997). We will refer to such capacities as JP-capacities. A *JP-capacity* is a convex combination of a convex capacity and its conjugate, which is a concave capacity, i.e., $\nu:=\alpha\mu+(1-\alpha)\overline{\mu}$ where μ is a convex capacity and $\alpha\in[0,1]$.

A GNAC $v(.|\mathcal{N},\pi,a,b)$ with the parameter restrictions $a\geq 0$ and $a+b\leq 1$, is a convex combination

$$v(\cdot|\mathcal{N},\pi,a,b)=\frac{(1-a-b)}{(1-b)}\mu(\cdot|\mathcal{N},\pi,b)+\frac{a}{(1-b)}\bar{\mu}(\cdot|\mathcal{N},\pi,b).$$

of the convex capacity $\mu(E|\mathcal{N},\pi,b)$,

$$\mu(E|\mathcal{N},\pi,b) := \begin{cases} 0 & \text{if } E \in \mathcal{N} \\ b\pi(E) & \text{if } E \notin \mathcal{N} \text{ and } E^c \notin \mathcal{N} \\ 1 & \text{if } E^c \in \mathcal{N} \end{cases},$$

and its concave conjugate, defined by $\bar{\mu}(E|\mathcal{N},\pi,b) = (1-b) + b\pi(E)$ for essential events E, $E^c \notin \mathcal{N}$.

Since μ is a convex capacity and $\bar{\mu}$ is a concave capacity, we can combine the arguments of the previous two subsections to obtain

$$CEU(f, v_E) = \frac{(1-a-b)}{(1-b)} \min_{p\in P_E} \int (u \circ f)\, dp + \frac{a}{(1-b)} \max_{p\in P_E} \int (u \circ f)\, dp, \tag{4}$$

where

$$P_E = C(\mu)_E = \bar{C}(\bar{\mu})_E .$$

A nice property of the FBU of a cavex GNAC is the fact that the weights given to the minimual and maximal expected utility in Equation 4, ($(1-a-b)/(1-b)$ and $a/(1-b)$) are independent of the event on which the cavex GNAC is updated. In terms of Figure 1 the transformation of π will only change in slope but not in its fixed point. Eichberger, Grant, and Kelsey (2009) provide a more detailed analysis of this property.

4.4 'Vexcave' GNAC $(a \leq 0, a+b \geq 1)$.

This parameter constellation of a GNAC is almost in contradiction with the monotonicity requirement of a capacity. In order to see this let $\underline{\pi} := \min_{E\notin\mathcal{N}} \pi(E)$ and consider the inequalities $\min_{E\notin\mathcal{N}} [a + b\pi(E)] = a + b\underline{\pi} \geq 0$ and $\max_{E\notin\mathcal{N}} [a + b(1-\pi(E))] = a + b(1-\underline{\pi}) \leq 1$. They imply $0 \geq a \geq -b\underline{\pi}$ and $1 \leq a+b \leq 1+b\underline{\pi}$. For an atomless state space, these inequalities would force the GNAC to equal its additive part.

Though conceptually possible, this parameter constellation appears difficult to reconcile with observable attitudes towards ambiguity.

5 Conclusion

In this paper, we show that a decision maker with CEU preferences satisfying *Consequentialism*, *State Independence,* and *Conditional Certainty Equivalent Consistency* will hold beliefs which are a linear transformation of an additive probability distribution. In a CEU-model of decision making under ambiguity consistency requirements between unconditional and conditional preferences restrict the class of capacities considerably. The class of capacities determined by these three axioms almost coincides with neo-additive capacities. CEU preferences with neo-additive capacities can be represented by as a linear combination of the expected utility with respect to some additive probability distribution and the maximum and minimum utility over outcomes. These three axioms also imply that the capacity of a CEU preference order must be updated according to the Full Bayesian Updating rule. If beliefs are represented by a convex capacity then the core of the Full Bayesian updated capacities equals the set of Bayesian updates of the probabilities in the core of the prior capacity. These observations clarify some open questions on Fully Bayesian updating of capacities and multiple priors and provide additional arguments for generalised neo-additive capacities in a dynamic context.

Appendix A. Proofs

For the proof of Proposition 3.1 we will use the following two properties of capacities which characterise GNACs. These properties were introduced in Proposition 3.1 of Chateauneuf, Eichberger, and Grant (2007).[8]

Property A $v(E \cup F) - v(F) = v(E \cup G) - v(G)$ is satisfied for all events E, F and G such that $v(F) \neq 0$, $v(F \cup E) \neq 1$, $v(G) \neq 0$ and $v(G \cup E) \neq 1$.

Property A characterises capacities which have identical increments for essential events. For such capacities the Choquet integral will have the same additive probability distribution for all rank-orders of states with the same best and worst state.

Property D Let N be a null event and E an essential event, then $v(N \cup E) = v(E)$.

Property D characterises capacities for which the union with a null event will not affect the capacity value of an essential event.

[8] Property A corresponds to (a) and Property D to (d) in Proposition 3.1 of Chateauneuf, Eichberger, and Grant (2007).

In the proof of Proposition 3.1 we will use the following characterisation of GNACs.

Lemma A.1 *Suppose that a capacity v satisfies property D, then the following assertions are equivalent:*
(i) v is a GNAC,
(ii) v satisfies Properties A.

Proof. We refer to the proof of Proposition 3.1 in Chateauneuf, Eichberger, and Grant (2007), pp. 556-559. They prove that a capacity v, which satisfies Properties A and D and two further properties, Properties (b) and (c), is a null-additive set function of the form $v(E) = \lambda + (1 - \delta)\pi(E)$ for any essential set E, where π is a probability distribution on S and $\lambda, \delta \in [0, 1]$ are real numbers. A careful reading of their proof reveals that properties (b) and (c) are only used in Part (b2) of their proof in order to establish $\lambda \leq 1$ and in Part (c) in order to show that $\delta \in [0, 1]$. Therefore, if only Property A and Property D are satisfied, then the capacity has no bounds on λ and δ, except the ones implied by monotonicity. It follows that a capacity ν satisfying Properties A and D is a GNAC. ■

Proof of Proposition 3.1:
(i) We suppose that CCEC is fulfilled and show that Properties A holds for a capacity ν satisfying Axioms 0 and CCEC.
For ease of notation we write $\int u(f)dv$ for $CEU(f, v)$ and $\int_E u(f)dv$ for $CEU(f \cdot E, v)$ where E is the indicator function of E. From Theorem 1 in Eichberger, Grant, and Kelsey (2007), we obtain that the von Neumann-Morgenstern utility u can be chosen to be independent of E.
Step A: Let $f = \sum\limits_{j=1}^{n} x_j A_j$ and $u(x_j) < u(x_{j+1})$ for all $j, 1 \leq j \leq n$. Note that there is an additive measure $m_{[f]}$ such that $v(A_j \cup A_{j+1} \cup ... \cup A_n) = m_{[f]}(A_j \cup A_{j+1} \cup ... \cup A_n) = \sum\limits_{i=j}^{n} m_{[f]}(A_i)$ and $\int u(f)dv = \int u(f)dm_{[f]}$. For any event A_i, denote by $E_i := A_i^c$. Then there is an (additive) measure $p_{[f]}$ such that $v_{E_i}(A_j \cup A_{j+1} \cup ... \cup A_n) = p_{[f]}(A_j \cup A_{j+1} \cup ... \cup A_n) = \sum\limits_{i=j}^{n} p_{[f]}(A_i)$ for $j \neq i, 1 \leq j \leq n$,and $\int u(f)dv_{E_i} = \int u(f)dp_{[f]}$.

In this step we will show that for any event $A_i,\ i \neq 1, n$, on which the act f takes no extreme value, the Choquet integral of f conditional on the information that the event A_i did not occur can be calculated according to the measure $m_{[f]}$ updated by the event $E_i = A_i^c$ according to Bayes rule $p_{[f]} = \frac{m_{[f]}}{m_{[f]}(E_i)}$.

Lemma A.2 *If v satisfies Axiom CCEC, then for any A_i with $i \neq 1, n$, and associated measures $m_{[f]}$ and $p_{[f]}$ such that $\int u(f)dv = \int u(f)dm_{[f]}$ and $\int u(f)dv_{E_i} = \int u(f)dp_{[f]}$, we have $p_{[f]} = \frac{m_{[f]}}{m_{[f]}(E_i)}$.*

Proof. Let y be the certainty equivalent of f conditional on $E_i,\ f \sim_{E_i} y$. By axiom CCEC, $f_{E_i}y \sim y$. If f and $f_{E_i}y$ are not comonotone, then we can modify the best and worst outcomes of the act $f_{E_i}y$ to obtain an act $g_{E_i}y$ such that f and $g_{E_i}y$ are comonotone and $g_{E_i}y \sim y$. Hence, $\int u(g)dv_{E_i} = \int u(g)dp_{[f]}$ As f and $g_{E_i}y$ are comonotone, their Choquet integrals are computed according to the same measure $m_{[f]}$, namely $\int u(g_E y)dv = \int u(g_E y)dm_{[f]}$. From $g_{E_i}y \sim y$, we get $u(y) = \int u(g_E y)dv = \int u(g_E y)dm_{[f]} = \int_{E_i} u(g)dm_{[f]} + m_{[f]}(A_i)u(y)$, which can be transformed to yield

$$u(y) = \frac{1}{m(E_i)} \int_{E_i} u(g)dm_{[f]} = \int_{E_i} u(g)dv_{E_i} = \int_{E_i} u(g)dp_{[f]}.$$

Let $\pi := \frac{m_{[f]}}{m_{[f]}(E_i)} - p_{[f]}$, we have $\int_{E_i} u(g)d\pi = 0$. Let us prove $\pi = 0$. For any act f' which is comonotonic with f we can construct g' and get $\int_{E_i} u(g')d\pi = 0$. Hence, $\pi = 0$. $\square$

Step B. From Lemma A.2, we have $p_{[f]} = \frac{m_{[f]}}{m_{[f]}(E_i)}$ for each act f and each measure $m_{[f]}$ such that $\int u(f)dv = \int u(f)dm_{[f]}$ whenever there is no extreme outcome of f on A_i. Hence, by varying the best and worst outcomes, one can find two *comonotone* acts f and f' such that $x_i = f(A_i)$ and $x'_i = f'(A_i)$ satisfy $u(x_j) < u(x_i) < u(x_{j+1})$ and $u(x_{j'}) < u(x'_i) < u(x_{j'+1})$ for $j \neq j'$. For the associated measures $m_{[f]}$ and $m'_{[f]}$, we have

$$\frac{m_{[f]}}{m_{[f]}(E_i)} = \frac{m'_{[f]}}{m'_{[f]}(E_i)}.$$

>From $m_{[f]}(E_i) = m'_{[f]}(E_i)$, we have $m_{[f]}(E_i) = 1 - m_{[f]}(A_i) = 1 - v(A_i \cup A_{j+1}...A_n) +$

$v(A_{j+1}...A_n)$, and $m'_{[f]}(E_i) = 1 - m'_{[f]}(A_i) = 1 - v(A_i \cup A_{j'+1}...A_n) + v(A_{j'+1}...A_n)$ or $v(A_i \cup A_{j+1} \cup ... \cup A_n) - v(A_{j+1} \cup ... \cup A_n) = v(A_i \cup A_{j'+1} \cup ... \cup A_n) - v(A_{j'+1} \cup ... \cup A_n)$.

This is true for any f. Hence, for $i \neq 1, n$, let $A_i = E$, $F = A_{j+1} \cup ... \cup A_n$ and $G = A_{j'+1} \cup ... \cup A_n$. Clearly, $E \cap F = \varnothing = E \cap G$. The left-hand side of the equality holds if $v(A_i \cup A_{j+1} \cup ... \cup A_n) - v(A_{j+1} \cup ... \cup A_n) \neq 1$, i.e. $v(F) \neq 0$ and $v(F \cup E) \neq 1$ (which insures us that v_E exists), the right-hand side of the equality holds for every G such that $v(G) \neq 0$ and $v(G \cup E) \neq 1$ (which insures us that $m(E) \neq 0$). Hence, we get

$$v(F \cup E) - v(F) = v(G \cup E) - v(G).$$

(ii) Let us suppose that property A is satisfied, we partition S in two sets U and U^c (this is possible because of the structure of the null sets) with $v(E) \neq 0$ for all non empty E included in U and $v(F) = 0$ for all $F \subset U^c$, let us note that for all $E \subsetneq U$, $v(E) < 1$, the atoms of U are A_i and the ones of N are B_j. we suppose that there are at least three atoms in U.

Let $A_i \subsetneq E \subsetneq U$, $N \subset U^c$. As the complement of $E \cup N$ is not included in U^c, then $0 < v(E \cup N) < 1$.We make use of property A:

$$(a) : v(E \cup N) - v(E \cup N \backslash A_i) = v(E) - v(E \backslash A_i)$$

>From (a) we draw two consequences:

1) If $v(A_1 \cup N) = v(A_1) + e$ then for all i, let $E = A_1 \cup A_i$, we get $v(A_i \cup N) = v(A_i) + e$ (nb that is where we use there are more than three atoms in U)

2) for all E, with $0 < v(E) < 1$, If $v(A_1 \cup N) = v(A_1) + e$ then $v(E \cup N) = v(E) + e$. This can be proved using $A_i \subset E$ and then applying 1) and (a).

Let $\epsilon_j = v(A_1 \cup B_j) - v(A_1)$. We can distinguish two cases:

- $\epsilon_j = 0$ in which case B_j plays no role,
- $\epsilon_j \neq 0$ in which case we have $v(B_j) = 0$ but $v(E \cup B_j) = v(E) + \epsilon_j$.

Consider now the capacity v/U (the restriction of v to U). It has no other null set than the empty one and fulfills property A. Therefore we can apply Lemma **??** and conclude that v/U is a GNAC, so for all $E \subset U$, $v/U(E) = a' + b'\pi'(E)$.Let $b = \sum_{i \in I} b'\pi'(A_i) + \sum_{j \in J} \epsilon_j$ and $\pi(A_i) = \frac{b'\pi'(A_i)}{b}$ and $\pi(B_j) = \frac{\epsilon_j}{b}$. We have then

$v(E) = 0$ if $E \subset U^c$,

$v(E) = 1$ if $U \subset E$,

$v(E) = a + b\pi(E)$ otherwise.

(iii) If ν is a GNAC then property A is directly satisfied.

(iv) Let us suppose that property A and is satisfied, that the FBU is used as the updating rule, and show that CCEC is satisfied. Let us consider an act $f_E x$ such that x is not an extreme value of the act $f_E x$. Let us say that $u(x_{i_0}) < u(x) < u(x_{i_0+1})$

Let $\int f_E x dv = \int f_E x dm_{[f_E x]}$, so $m_{[f_E x]}(E) = 1 - m_{[f_E x]}(E^c) = 1 - v(E \cup A_{i_0+1} \cup ... \cup A_n) + v(A_{i_0+1} \cup ... \cup A_n)$.

$$\int f dv_E = \sum u(x_i)(v_E(A_i \cup A_{i+1} \cup ... \cup A_n) - v_E(A_{i+1} \cup ... \cup A_n))$$

As $v_E(A) = \frac{v(A)}{v(A)+1-v(E^c \cup A)}$, and by property A $v(E \cup A_{i_0+1} \cup ... \cup A_n) - v(A_{i_0+1} \cup ... \cup A_n) = v(E^c \cup A) - v(A)$, which implies $v(A) + 1 - v(E^c \cup A) = m_{[f_E x]}(E)$, we have:

$$\int f dv_E = \sum u(x_i) \frac{v(A_i \cup A_{i+1} \cup ... \cup A_n) - v(A_{i+1} \cup ... \cup A_n)}{m_{[f_E x]}(E)}$$

Now we must distinguish two cases:

- $u(x) < u(x_i)$, then $m_{[f_E x]}(A_i) = v(A_i \cup A_{i+1} \cup ... \cup A_n) - v(A_{i+1} \cup ... \cup A_n)$,

- $u(x) > u(x_i)$ then $m_{[f_E x]}(A_i) = v(A_i \cup A_{i+1} \cup ... \cup A_n \cup E) - v(A_{i+1} \cup ... \cup A_n \cup E)$, by property A,

$v(A_i \cup A_{i+1} \cup ... \cup A_n \cup E) - v(A_{i+1} \cup ... \cup A_n \cup E) = v(A_i \cup A_{i+1} \cup ... \cup A_n) - v(A_{i+1} \cup ... \cup A_n)$

Therefore

$$\int f dv_E = \frac{\int_E f dm_{[f_E x]}}{m_{[f_E x]}(E)}$$

$f_E x \sim x \Leftrightarrow \int f_E x dv_E = u(x) \Leftrightarrow \int f_E x dm_{[f_E x]} = u(x) \Leftrightarrow \int_E f dm_{[f_E x]} + u(x) m_{[f_E x]}(E^c) = u(x) \Leftrightarrow \int_E f dm_{[f_E x]} = u(x) m_{[f_E x]}(E) \Leftrightarrow \frac{\int_E f dm_{[f_E x]}}{m_{[f_E x]}(E)} = u(x) \Leftrightarrow \int f dv_E \Leftrightarrow f \sim_E x$

So CCEC holds.

■

Proof of Proposition 4.1:

$C(v_E)$ is the core of a convex capacity. It is known, see e.g. Delbaen (1974), that for any maximal chain (a chain is an ordered set of sets) $C_1 \subset ..C_i.. \subset E$ there exists $\mu \in C(v_E)$ such that $\forall i\ \mu(C_i) = v_E(C_i)$. $\mu \in P_E$ so for all i there exists $p \in C(v)$ such that,

$$\frac{p(C_i)}{p(E)} = v_E(C_i) = \frac{v(C_i)}{v(C_i) + \overline{v}(E \backslash C_i)}$$

It thus follows from computations made above that $p(C_i)\overline{v}(E \backslash C_i) - v(C_i)p(E \backslash C_i) = 0$. As $p(C_i) \geq v(C_i)$ and $p(E \backslash C_i) \leq \overline{v}(E \backslash C_i)$, we get $p(C_i) = v(C_i)$ and $p(E \backslash C_i) = \overline{v}(E \backslash C_i)$. From (1) we deduce that for all i,

$$p(E) = v(C_i) + \overline{v}(E \backslash C_i) = 1 + v(C_i) - v(C_i \cup E^c)$$

so for A and B non void strictly included in E and ordered by inclusion we have,

$$v(E^c \cup A) - v(A) = v(E^c \cup B) - v(B)$$

We can prove it remains true if A and B are not ordered by inclusion. if $A \cap B \neq \emptyset$, we have,

$$v(E^c \cup A) - v(A) = v(E^c \cup (A \cap B)) - v(A \cap B) = v(E^c \cup B) - v(B)$$

if $A \cup B \subsetneq E$ we do the same with $A \cup B$:

$$v(E^c \cup A) - v(A) = v(E^c \cup (A \cup B)) - v(A \cup B) = v(E^c \cup B) - v(B)$$

The remaining case is $A \cup B = E$ and $A \cap B = \emptyset$, if $|E| > 2$, we pick a non void set included in A or B, say $A\acute{}$, and get,

$$v(E^c \cup A) - v(A) = v(E^c \cup A\acute{}) - v(A\acute{}) = v(E^c \cup (A\acute{} \cup B)) - v(A\acute{} \cup B) = v(E^c \cup B) - v(B)$$

if $|E| = 2$, as $|S| > 3$ we can write $E^c = F \cup G$ and get,

$$(i): \quad v(F \cup G \cup A)) - v(G \cup A) = v(F \cup G \cup B) - v(G \cup B)$$

$$(ii): \quad v(G \cup A) - v(A) = v(G \cup B) - v(B)$$

$$(i) - (ii): \quad v(E^c \cup A)) - v(A) = v(E^c \cup B) - v(B)$$

So we get the property A which insures that v is a GNAC.

Conversely, let us suppose that v is a GNAC, we just need to prove that any extreme point μ of $C(v_E)$ belongs to P_E. There exits a maximal chain $C_1 \subset ..C_i..C_k \subset E$ such that $\forall i$, $\mu(C_i) = v_E(C_i)$. We are going to construct $p \in C(v)$ such that for all i,

$$\frac{p(C_i)}{p(E)} = v_E(C_i) = \frac{v(C_i)}{v(C_i) + \overline{v}(E \backslash C_i)}$$

On $\mathcal{P}(E)$, the set of parts of E, $v_{/E}$, v restricted to E is a convex capacity, so we can find in its core a probability p such that $p(C_i) = v(C_i)$ and $p(E) = v(C_k) + \overline{v}(E \backslash C_k)$, (compare Delbaen (1974)). By the Hahn Banach Theorem, we can extend p to Σ with p in the core of v. As v satisfies property A we have

$$p(E) = v(C_i) + \overline{v}(E \backslash C_i) = 1 + v(C_i) - v(C_i \cup E^c).$$

Hence,

$$p(C_i \cup E^c) = p(E) + p(C_i) = v(C_i \cup E^c).$$

Thus, p satisfies property A and we have $C(v_E) = P_E$. ■

Proof of Proposition 4.2:

A *chain* is a collection of sets ordered with respect to inclusion. It is maximal when it is maximal for inclusion, i.e., if adding a set to the collection makes it no longer an ordered collection. We make use of the following result (Delbaen (1974), pp. 219-220): C is the core of a convex capacity if and only if for all maximal chains (A_i) there exists $m \in C$ such that $m(A_i) = \min_{p \in C} p(A_i)$. These measures m are the the extreme points of C. Hence, $C = \overline{\text{co}}\{m \in C |$ there exists a maximal chain such that $m(A_i) = \min_{p \in C} p(A_i)\}$.

We have to prove $C(v_E) \subset P_E$. Let (A_i) be a maximal chain of E and $\mu \in C(v_E)$ such that $\mu(A_i) = \min_{p \in C(v_E)} p(A_i)$. We want to show that $\mu \in P_E$.

(i) Consider any additive measure m such that $m(A) = v(A)$ and $m(E) = 1 - v(E \backslash A) + v(A)$

and let $p \in C(v)$, then

$$\frac{p(A)}{p(E)} - \frac{v(A)}{1 - v(E^c \cup A) + v(A)} = \frac{p(A)(1 - v(E^c \cup A) + v(A)) - v(A)(p(E\backslash A) + p(A))}{(1 - v(E^c \cup A) + v(A))p(E)}$$
$$= \frac{p(A)(1 - v(E^c \cup A)) - v(A)(p(E\backslash A))}{(1 - v(E^c \cup A) + v(A))p(E)} \geq 0,$$

since every $p \in C(v)$ satisfies $p(A) \geq v(A)$ and $p(E\backslash A) \leq 1 - v(E^c \cup A)$. Hence,

$$\frac{p(A)}{p(E)} \geq \frac{v(A)}{1 - v(E^c \cup A) + v(A)}$$

(ii) As v is convex for any chain there exists a measure in its core which is equal to the capacity for each element of the chain. Hence, for the chain $A, A \cup E^c, S$ there exists $m \in C(v)$ such that $m(A) = v(A)$ and $m(E\backslash A) = 1 - v(E\backslash A)$. As the set P_E is the core of a convex capacity, for any maximal chain (A_i) of E there exists a measure m such that $m(A_i) = \min\limits_{p \in P_E} p(A_i)$. From (i), m satisfies $m(A_i) = v(A_i)$ and $m(E) = 1 - v(E\backslash A_i) + v(A_i)$ for all i.

Therefore $m = \mu$ and v is a GNAC according to Proposition 4.1.

Proof of Remark 3.1

Let us check that GNAC satisfy this axiom: let $\arg\min\limits_{s \in S} f(s) \cap \arg\min\limits_{s \in S} g(s) = E_m$ and $\max\limits_{s \in S} f(s) \cap \arg\max\limits_{s \in S} g(s) = E_M$. Let $p = (1-\delta)\pi + \alpha\delta d_{E_m} + (1-\alpha)\delta d_{E_M}$, where d_E denotes the Dirac measure of the set E. As $\max\left\{\min\limits_{s \in S} f(s), \min\limits_{s \in S} g(s)\right\} \leq \min\limits_{s \in S} h(s), \quad \max\limits_{s \in S^c} h(s) \leq \min\left\{\max\limits_{s \in S} f(s), \max\limits_{s \in S} g(s)\right\}$ then $\int f_A h dv = \int f_A h dp$ and $\int g_A h dv = \int g_A h dp$. As E_m and E_M are included in A, then $\int f dv_A = \int f dp_A$ and $\int g dv_A = \int g dp_A$. Therefore CCEC comes from that the same measure at every stage. ■

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