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Abstract

Least-Unmatched Price Auctions have become a popular format of TV and radio shows. Increasingly, they are also applied in internet trading. In these auctions the lowest single (unique) bid wins. We analyze the game-theoretic solution of least unmatched price auctions when prize, bidding cost and the number of participants are known. We use a large data-set of such auctions in order to contrast actual behavior of players with game-theoretic predictions. In the aggregate, bidding behaviour seems to conform with a Nash equilibrium in mixed strategies.

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1 Introduction

In standard auctions, the highest bid wins. English auctions and first- and second-price sealed-bid auctions are prime examples. Recently, new forms of auctions based on the opposite principle became popular in different areas. In *"reverse auctions"* or *"backward auctions"*, as they are sometimes called, the lowest bid wins. Without further restrictions this rule would simply induce all bidders to bunch on the lowest price. The additional rule that only an unmatched (single) bid can win, however, forces participants to predict the bids of other participants. It is this prediction problem, which makes the *L*east-*U*nmatched *P*rice *A*uction (*LUPA*[1]) worth studying from a strategic point of view.

According to a report in *USA Today* (25th of October 2006), prizes worth $360,000 were won by bids totalling less than $1,000 in the first 250 LUPAs run in the USA . The UK-based company *Auction Air Ltd.* organized more than 500 LUPAs. Since fall 2004, it allocated prizes totalling more than $700,000 to winning bids worth about $12,000. Least Unmatched Price Auctions may appear an odd trading mechanism. Indeed, they are often used as a marketing instrument for TV- and radio shows. In fall 2005, when fuel prices sky-rocketed, the German radio station *Radio Brocken* sold petrol vouchers worth €500 in a daily LUPA. Also in 2004, the London radio station *Capital FM Radio* sold a flat in London, a house in Spain, and a Bentley Continental in several LUPAs. Several other LUPAs were run on German radio and television in 2005-06.

In recent years, game shows on radio and television have become a fascinating area of economic research. The data of some shows has been used for the analysis of the participants' behavior. Already in 1993, Gertner (1993) analyzed a game show called *"Card Sharks"*. More recently, Hartley, Lanot and Walker (2005) and Post et. al (2006) studied such popular TV shows as *"Who Wants to be a Millionaire?"* and *"Deal or No Deal?"*. Data from these shows allow these authors to investigate whether participants' behavior was consistent with the economic notion of rationality and the degree of risk-aversion. Analyzing the show *"The Price Is Right"*, Berk et al. (1996) conclude that behavior of contestants is not fully rational, whilst Tenorio and Cason (2002) come to the opposite conclusion that contestants are quite capable of making optimal decisions even in difficult situations.

In the game show *"Jeopardy!"*, players choose the category and difficulty of questions in

[1] One can also read this acronym as "lowest unique price auction".

order to maximize their own chances for giving the right answer while making it more difficult for their rivals. With data from this show METRICK (1995) and BOYLE AND SHAPIRA (2006) can highlight behavioral aspects of decision-making. ANTONOVICS, ARCIDIACONO, WALSH (2005) study *"The Weakest Link"*, a game show in which players gain money for a coalition by answering questions and secure these gains for themselves by voting to exclude other players from the coalition, in order to address the issues of gender discrimination. While chance plays an important role in all the games mentioned so far, BOSCH-DOMENECH ET AL. (2002) study beauty contests, organized through newspapers, which are purely strategic games.

Game shows provide natural experiments for studying the behavior of large numbers of participants. Moreover, they often involve high gains and, thus, offer stronger incentives. Similar to internet auctions such as Ebay and Amazon, which were studied by OCKENFELS AND ROTH (2002, 2006) and others, radio and TV shows are also field experiments (LUCKING-REILEY 1999). Thus, they meet two major points of critique advanced against laboratory experiments: small prizes and small numbers of participants. BOSCH-DOMENECH ET AL. (2002) compare beauty contest experiments conducted in laboratory settings with those run in newspapers. Their studies suggest that some behavioral patterns observed in laboratory experiments with small numbers of participants may disappear in field experiments with larger pools of players. On the other hand, TENORIO AND CASON (2002) can replicate the results obtained in a field experiment on TV in a laboratory experiment. They show that the behavior of people in natural experiments with large stakes may not differ significantly from behavior observed in laboratory experiments with much smaller monetary stakes.

LUPAs are a special case of unmatched bid auctions which have been studied also by other authors. DE WACHTER AND NORMAN (2006) run laboratory experiments with *"minbid auctions"*. They consider the case where players are restricted to only one bid and compare the results from their laboratory experiment with a Monte Carlo simulation. RAPPOPORT ET AL. (2007) consider high and low *"unique bid auctions"* where bidders are also restricted to a single bid. They provide a numerical approximation of the solution for a game-theoretic model and compare it with the results of a laboratory experiment. ÖSTLING, ET AL. (2007) run a *"lowest unique positive integer"* experiment and contrast the observed behavior with the solution of a Poisson game with a single bid per player.

The LUPAs which we study in this paper are field experiments with mostly large prizes involving large numbers of participants (sometimes tens and hundreds of thousands). In some cases we have a relatively small number of participants (one or two hundreds players), in others there is a five hundred times larger number of participants. Prizes in these LUPAs vary from \$200 to \$500,000. These LUPAs have been run under a variety of conditions and in several countries, providing us with an opportunity to verify the robustness of our theoretical predictions. Moreover, for some of these LUPAs we have obtained a detailed data at the micro-level[2]. To the best of our knowledge, there are no other examples of experiments with such large numbers of participants and such large stakes.

In this paper, we model LUPAs as non-cooperative games where, as in the field experiments, players can place multiple bids. We show that large LUPAs have no equilibrium in pure strategies. We use the data from several LUPAs to characterize the symmetric equilibrium in mixed strategies. For the LUPAs with a large number of players, we find that aggregate bidding is well described by a symmetric Nash equilibrium in mixed strategies. Individual bidders, however, almost never follow the equilibrium strategy. Consistent with the theoretical result, people tend to place lower bids more frequently than higher bids. Yet the actual frequency of low bids is higher than theoretically predicted. We also find that the actual number of active participants in LUPAs is much higher than predicted by the game-theoretic model.

In Section 2 we describe the game. The Nash equilibrium is discussed in Section 3 and, in Section 4, we present the data from several least unmatched price auctions with varying number of players, prizes and bidding costs. We conclude with a discussion of our results and some comments about further research.

2 Least unmatched price auctions

The rules of a least unmatched price auction are as follows. The organizer of the game announces the item to be sold, and the period within which bidding may take place (bidding phase). Bids must be submitted in local currency, say in euros and cents. Bids in non-integer amounts of euros and cents are not accepted. Agents who wish to take part in the game, place their bids via a phone call or an SMS. The number of bids which an agent can submit is not restricted. For each bid a fixed cost is charged, which is included in the cost of calling or sending an SMS. No information about bids is provided during the bidding phase of the game.

As soon as the bidding phase is over, the winner is determined from the set of valid bids submitted. The winning bid must satisfy two criteria:

[2] In these cases, we gratefully acknowledge the support of Legion Telekommunkation GmbH and Radio Regenbogen.

1. It must be unmatched, i.e. there is no other player who has placed the same bid.
2. It must be the lowest bid among all unmatched bids.

The winner is the player, who made the winning bid. The winner pays the winning bid to the auctioneer and receives the prize.

Table 1 summarizes some information about LUPAs, which took place in Germany in 2005-2006. They had different formats and were run on radio, TV and in newspapers. Participants could bid through different channels: SMS, phone call, internet, or voucher. In all auctions, providers charged bidders with 0,49 cents per bid. The auctions had fixed duration and a variable number of bidders, both shown in the table. There was no restriction on the number of bids per bidder.

Media	**Prize**	**Cost**	**Number of bidders**	**Total Bids**	**Duration**	**Winning bid**
Radio	10000€ monetary	0,49€	9400	47872	19 days	14,55€
Radio	10000€ monetary	0,49€	10660	52847	8 days	14,65€
Radio	1000€ monetary	0,49€	537	1798	2 days	0,60€
Radio	3000€ monetary	0,49€	916	6732	4 days	5,82€
Radio	5000€ monetary	0,49€	631	6201	5 days	11,16€
Newspaper	1099€ mountain bike	0,49€	437	1272	17 days	1,51€
TV	20 000€ tuned car	0,49€	89862	266824	7 days	20,65€
Radio	350 000€ House	0,49€	72588	610104	23 days	99,82€

Table 1. Summary of some LUPAs run in Germany in 2005-2006

Table 2 summarizes data from some recent LUPAs organized by AuctionAir$^{\text{TM}}$. All auctions were run online: costs were charged to the participant's credit card after online registration. In all auctions a maximal number of bidders was specified (see "Bids required" in the table). Once this number was reached, the auction was closed. Therefore, the duration of the auction was unknown for participants, though they were informed about the number of bidders. Auctions were repeated: the same item was auctioned off several times under the same conditions. We indicate the number of preceding auctions in the table. Data from preceding auctions was available to bidders. The table provides also the winning bids of the last auction. Bidders were restricted in the number of bids they could place (see "Max bids per pers."). This constraint was, however, mostly not binding. Identification of a player was made by the credit card number. Hence, using several cards or building up coalitions of bidders (e.g. family members), the number of bids per person was effectively unrestricted.

Prize	**Cost**	**Bids required**	**Max bids per pers.**	**Preceding auctions**	**Winner**
£259 80 GB iPod	£3.00	120	10	6	£24.00
£1,695 40" LCD TV	£4.00	575	20	28	£6.00
£5,900 7 nights in Mauritius	£12.00	530	20	2	£49.00
£275 Headphones	£5.00	60	5	39	£12.00
£16,900 Mini Cooper	£25.00	945	20	11	£20.00

Table 2. Summary of some recent LUPAs run by AuctionAir.com

2.1 Formal description

Let us denote by $I = \{1, .., N\}$ the set of potential bidders. We will assume that bids are denominated in cents, i.e., a bid of \$12.34 corresponds to the number 1234, etc. Hence, we can identify the set of bids with the set of natural numbers $\mathbb{N}$. During the bidding phase, each player i can place an arbitrary number of bids $b_i \in \mathbb{N}$. There exists a bidding cost of c cents per bid. If player i wins, he obtains the prize. We denote the value of the prize by A. Though a participant could make bids sequentially, we can treat his strategy s_i as the simultaneous choice of a set of bids $s_i = \left\{b_i^1, b_i^2, ..\right\}$, provided that no information about other participants' behavior is released during the bidding phase. Let $s^0 = \{\varnothing\}$ denote the outside option of not bidding at all. The following result follows from the fact that the best possible outcome for player i is to win A with a single bid b_i, at cost c.

Proposition 2.1 *Any strategy s_i containing bid $b_i > A - c$ is strictly dominated by strategy s^0.*

Without loss of generality, this result allows us to restrict the set of bids to an interval of natural numbers $b = 1..\mathbf{b}$, with $\mathbf{b} = A - c$. Any pure strategy can be represented by a binary vector, e.g. (1,0,0,1,0..,0) with "1" at position b meaning that the player places bid b, and "0" meaning that bid b is not placed according to this strategy. Since bids b are ordered as natural numbers, the representation of each pure strategy is unique. Each pure strategy s^t may be assigned number $\nu\left(s^t\right) = t$ such that the reverse binary representation of this number corresponds to the unique combination of zeros and ones in that strategy. Pure strategy s^0 corresponds to the option of not entering the game. Pure strategy s^1 has the form (1,0,0..0) and corresponds to only placing bid $b = 1$. Pure strategy s^5 has the form (1,0,1,0..0) and corresponds to placing two bids: $b = 1$ and $b = 3$. One can view the strategy $s^\nu = (1, 1, 1, 0..0)$ as the reverse of the binary number 0..0111, which corresponds to the number 7 in the decimal system and denote this strategy bys s^7. It is easy to check that there are in total $2^{\mathbf{b}}$ pure strategies in the strategy set of each player $S_i = \{s^t\}_{t=0}^{2^{\mathbf{b}}-1}$, including strategy s^0. For any bid b and any strategy $s_i = s^t$, we write $s_i(b) = s^t(b)$ for the binary number associated with b in strategy s^t, which equals one if this bid b is placed and or zero otherwise.

Denote with $S = \prod_{i \in I} S_i$ the set of all possible strategy combinations. Given a combination $s = (s_1, s_2, ..s_N)$ of pure strategies $s_i \in S$ for all players $i = 1..N$, the winning bid of the LUPA is determined.

Definition 2.1 *Bid $b = \mu(s)$ is **the least unmatched bid** if and only if the following two conditions are met:*

1. $\sum_{j=1}^{N} s_j(b) = 1,$
2. $\sum_{j=1}^{N} s_j(k) \neq 1,\ \forall k < b.$

The first condition means that bid b is unmatched, i.e., only one player places b in the given strategy combination. The second condition means that there are no unmatched bids k smaller than b, i.e., each bid $k < b$ is placed either more than once or not placed at all.

The payoff of player i if strategy combination s is played, is

$$p_i(s) = \begin{cases} A - \mu(s) - c\sum_{k=1}^{\mathbf{b}} s_i(k) & \text{if} \quad s_i(\mu(s)) = 1 \\ -c\sum_{k=1}^{\mathbf{b}} s_i(k) & \text{otherwise} \end{cases}.$$

Or, equivalently,

$$p_i(s) = (A - \mu(s))\, s_i(\mu(s)) - c\sum_{k=1}^{\mathbf{b}} s_i(k)\,. \tag{1}$$

If player i plays $s_i = s^0$ then $p_i(s^0, s_{-i}) = 0$ for all strategy combinations of his rivals s_{-i}. In our notation, we use theconvention of splitting a strategy combination s into strategy $s_i \in S_i$ played by player i and the opponents' strategy combination, $s_{-i} \in S_{-i} = S \setminus S_i$. Note that each player can guarantee himself a payoff of zero by not participating in the game, i.e., by choosing $s_i = s^0$. This fact has been used above for the elimination of dominated strategies.

This completes the description of the LUPAs as a game in strategic form.

2.2 Nash equilibria

A *Nash equilibrium* is defined as a strategy combination $s^* = (s_i^*, s_{-i}^*)$ such that for all $i \in I$,

$$p_i\left(s_i^*, s_{-i}^*\right) \geq p_i\left(s_i, s_{-i}^*\right), \forall s_i \in S_i.$$

Our first result shows that a Nash equilibrium in pure strategies exists only if bidding costs c are high relative to the prize A.

Proposition 2.2 1. *For $c > A - 1$, there exists a* unique *Nash equilibrium in pure strategies: $(s_1^*, ..., s_N^*)$ with $s_i^* = s^0$ for all $i \in I$.*

2. *For $c = A - 1$, there exist $(N + 1)$ Nash equilibria in pure strategies:*
   - *$(s_1^*, ..., s_N^*)$ with $s_i^* = s^0$ for all $i \in I$ and,*
   - *for any $i = 1, ..., N$, $(s_1^*, ..., s_N^*)$ with $s_i^* = s^1$ and $s_j^* = s^0$ for $j \neq i$.*
3. *For $c \in \left[\frac{A}{2} - 1; A - 1\right)$, there are N Nash equilibria in pure strategies: for any $i = 1, ..., N$, $(s_1^*, ..., s_N^*)$ with $s_i^* = s^1$ and $s_j^* = s^0$ for $j \neq i$.*
4. *For $c < \frac{A}{2} - 1$, there exist* no *Nash equilibria in pure strategies.*

The most interesting case is the one with low bidding cost. In all LUPAs which were played the bidding cost c was very small compared to the prize A. Hence, condition $c < \frac{A}{2} - 1$ is met and no equilibrium in pure strategies exists. Since the number of strategies in each player's strategy set S_i is finite, existence of a Nash equilibrium in mixed strategies follows immediately from Nash's theorem (Nash 1950)

2.3 Mixed strategies and expected payoff

Consider arbitrary player i. Let $\pi_i = \left\{\pi_i^t\right\}_{t=0}^{2^{\mathbf{b}}-1}$ be his mixed strategy, with $\pi_i^t = \pi_i\left(s^t\right)$ the probability of his playing strategy s^t. Consider a combination π of such mixed strategies for all players. Strategy combination $s \in S$ is played with probability

$$\pi(s) = \prod_{i=1}^{N} \pi_i(s_i).$$

With this notation, the expected payoff of player i may be written as

$$P_i(\pi_i, \pi_{-i}) = \sum_{s \in S} \pi(s) p_i(s)$$

The expected payoff of player i may be decomposed in the following way

$$P_i(\pi_i, \pi_{-i}) = \sum_{s \in S} \pi(s)(A - \mu(s)) s_i(\mu(s)) - c \sum_{s \in S} \pi(s) \sum_{k=1}^{\mathbf{b}} s_i(k). \tag{2}$$

The second term determines the expected costs of player i, and the first one his expected prize.

Lemma 2.3 *The expected payoff $P_i(\pi_i, \pi_{-i})$ of player i may be represented in a following way:*

$$P_i(\pi_i, \pi_{-i}) = \sum_{k=1}^{\mathbf{b}} (A-k) \sum_{s \in S} \pi(s)\, s_i(k) \rho(s,k)\, \psi(s,k) - c \sum_{t=0}^{2^{\mathbf{b}}-1} \sum_{k=1}^{\mathbf{b}} \pi_i^t s^t(,) \quad (3)$$

$$\textit{with } \rho(s,k) = \left[\sum_{j=1}^{N} s_j(k) \prod_{h \neq j} (1 - s_h(k)) \right] \quad (4)$$

$$\textit{and } \psi(s,k) = \prod_{l=1}^{k-1} \left[1 - \sum_{j=1}^{N} s_j(l) \prod_{h \neq j} (1 - s_h(l)) \right]. \quad (5)$$

The Boolean functions $\rho(s,k)$ and $\psi(s,k)$ in representation (3) determine whether bid k is unmatched in strategy combination s (function $\rho(s,k)$, equation 4) and whether there are no unmatched bids below k (function $\psi(s,k)$, equation 5).

3 Equilibrium

De Wachter and Norman (2006), Rappoport et al. (2007) and Östling et al. (2007) assume that players can only place a single bid. This is equivalent to restricting the strategy sets of the players to $\widetilde{S} = \left\{ s^{2^k} \right\}_{k=0}^{\mathbf{b}} \subset S$. This implies that players have no *"no entry"* option, i.e., not to bid at all. With this assumption the analysis of the game becomes very complicated and no closed-form solution has been suggested so far. In the approach suggested in this paper, multiple bids are feasible and there is the option of not participating in the bidding. These assumptions are not only a better description of the LUPAs as they were played in the field experiments, but we can also show that there is a closed-form solution, at least for a special case.

We consider only symmetric equilibria with $\pi_i^* = \pi^* = \left(\pi^{0*}, \pi^{1*}, ..\pi^{t*}, ..\pi^{(2^{\mathbf{b}}-1)*} \right)$ for all i.

In a first step, we find an explicit solution for a LUPA in which only strategies of a special form can be played. In a second step, we discuss the intuition behind such a reduction of the strategy set and show that this constraint is not always binding. In a third step, we give some numerical examples.

Proposition 3.1 *Consider a LUPA with prize A, bidding cost c and N players. Let players'* *strategy sets be restricted to* $\widehat{S} = \left\{ s^{2^k-1} \right\}_{k=0}^{\mathbf{b}} \subset S$. *Then* $\exists M < \mathbf{b} : \sum_{m=0}^{M-1} \frac{c}{A-(m+1)} \leq 1$ *and* $\sum_{m=0}^{M} \frac{c}{A-(m+1)} > 1$, *and*

$$\pi^{t*} = \begin{cases} \sqrt[N-1]{\sum_{m=0}^{k} \frac{c}{A-(m+1)}} - \sqrt[N-1]{\sum_{m=0}^{k-1} \frac{c}{A-(m+1)}} & \text{if} \quad t = 2^k - 1, \\ & \quad\quad 0 \leq k < M \\ 1 - \sqrt[N-1]{\sum_{m=0}^{M-1} \frac{c}{A-(m+1)}} & \text{if} \quad t = 2^M - 1 \\ 0 & \quad\quad \text{otherwise} \end{cases} \tag{6}$$

is a symmetric Nash equilibrium in mixed strategies.

The restriction to the strategy set to $\widehat{S}$ means that only strategies of the form $s^{2^k-1} = \left(\underbrace{1..1}_{k}, 0..0\right)$ can be played. This allows us to simplify the derivation of the probability that player i wins with a bid b, i.e.,the probability that bid b is unmatched *minus* the probability that this player wins with a lower bid. There is no chance that any other player wins with a lower bid. This reasoning is the central argument in the proof that leads to the explicit formula (6).

The examples below show that the class of LUPAs for which (6) is an unconstrained Nash equilibrium, is non-empty. On the other hand, there exist LUPAs, for which condition $s \in \widehat{S}$ is binding. Therefore, a restrictions of the strategy set to $\widehat{S}$ does not lead to a general Nash equilibrium but represents a constrained approximation of it. The extension of the constrained Nash equilibrium to the unconstrained case is left for future research.

Example 3.1 *Consider a LUPA with two players,* $N = 2$, *a prize* $A = 4$ *and bidding costs* $c = 1$. *This yields the following payoff matrix where only the payoffs of the row player* R *are shown:*

$R \setminus C$	s^0	s^1	s^2	s^3	s^4	s^5	s^6	s^7
s^0	0	0	0	0	0	0	0	0
s^1	2	−1	2	−1	2	−1	2	−1
s^2	1	−1	−1	−1	1	−1	−1	−1
s^3	1	0	1	−2	1	0	1	−2
s^4	0	−1	−1	−1	−1	−1	−1	−1
s^5	1	−1	1	−2	1	−2	1	−2
s^6	0	−2	−1	−2	0	−2	−2	−2
s^7	0	−1	0	−2	0	−1	0	−3

In this example, we have restricted the set of bids to $\{1, 2, 3\}$, *since bidding* 4 *is dominated by strategy* s^0. *This game has a unique symmetric equilibrium in mixed strategies* $\pi_R^* = \pi_C^* = \left(\frac{1}{3}, \frac{1}{2}, 0, \frac{1}{6}, 0, 0, 0, 0\right)$. *Two families of asymmetric equilibria include one player playing a pure strategy:* $\pi_i^* = (0, 1, 0, 0, 0, 0, 0, 0)$ *and the other the strategy* $\pi_{-i}^* = (\alpha, 0, 0, 1 - \alpha, 0, 0, 0, 0)$, $\alpha \in \left[\frac{1}{3}, 1\right]$, $i = R, C$. *Notice that condition* $c < \frac{A}{2} - 1$ *of proposition 2.2 is violated. The same game with bidding cost of* 0.9 *generates a unique equilibrium in mixed strategies* $\pi_R^* = \pi_C^* =$

$(0.3; 0.45; 0; 0.25; 0; 0; 0; 0)$.

Following example illustrates how the equilibrium changes, if the number of players increases from two to three.

Example 3.2 *Consider the same LUPA as above (with prize $A = 4$, and bidding cost $c = 1$) but with $N = 3$. In this case, the symmetric equilibrium is given by the following mixed strategy:*

$$\begin{aligned}
\pi^0 &= \sqrt{\frac{1}{3}} \approx 0.57735, \\
\pi^1 &= \sqrt{\frac{5}{6}} - \sqrt{\frac{2}{6}} \approx 0.33552, \\
\pi^2 &= 0 \\
\pi^3 &= 1 - \pi^0 - \pi^1 - \pi^2 \approx 0.087129.
\end{aligned}$$

Again, only the pure strategies s^0, s^1 and s^3 are played in equilibrium. All other strategies are played with probability zero.

Increasing the number of players further reveals a change in the equilibrium structure.

Example 3.3 *Consider the same LUPA with prize $A = 4$, bidding cost $c = 1$ as above, but increase the number of players from three to seven. The unique symmetric equilibrium in this game is*

$$\begin{aligned}
\pi^0 &= 6\pi^1 \approx 0.83, \\
\pi^1 &= \sqrt[6]{\frac{1}{2}\frac{1}{7^6 - 6^6}} \approx 0.1385, \\
\pi^2 &= \sqrt[6]{\frac{1}{3}} - \pi^0 \approx 0.00198, \\
\pi^3 &= 1 - \pi^0 - \pi^1 - \pi^2 \approx 0.0288,
\end{aligned}$$

Note that applying the solution of Proposition 3.1 would yield the following result:

$$\begin{aligned}
\pi^0 &= \sqrt[6]{\frac{1}{3}} \approx 0.8327, \\
\pi^1 &= \sqrt[6]{\frac{5}{6}} - \sqrt[6]{\frac{1}{3}} \approx 0.97 - 0.8327 = 0.1373, \\
\pi^2 &= 0, \\
\pi^3 &= 1 - \pi^0 - \pi^1 - \pi^2 \approx 0.03.
\end{aligned}$$

3.1 Equilibrium Distribution of Bids

Given the equilibrium in mixed strategies, we can find the probabilities with which individual

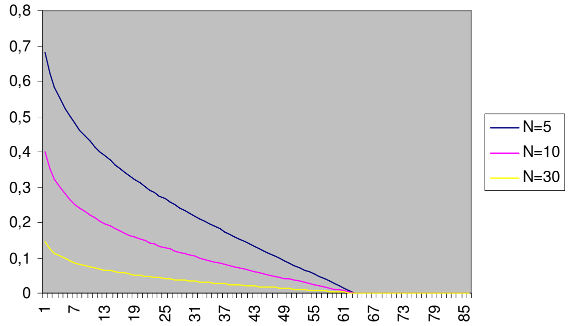

Figure 1. Probabilities of bids in three LUPAs with A=100, c=1, and number of players 5, 10, and 30.

bids are placed:

$$q^b = \sum_{t=0}^{2^b-1} \pi^{t*} s^t(b).$$

In a preliminary analysis, we use the structure of the equilibrium, described in Proposition 3.1 as an approximation of the "true" equilibrium. Considering strategies with numbers $t = \nu_k = 2^k - 1$. For these strategies , bid b is only placed in strategies with numbers $\nu_k \geq 2^b - 1$. Hence, bid $b = 1$ is placed in all strategies, which are played in equilibrium with positive probabilities, except strategy s^0.

The probability of bid $b = 1$ being placed in equilibrium is

$$q^1 = 1 - \pi^0 = 1 - \sqrt[N-1]{\frac{c}{A-1}}. \tag{7}$$

Note that example 3.3 yields the same probability for the bid $b = 1$ both for the precise and the approximated version of equilibrium.

Similarly, bid $b = 2$ is placed with probability

$$q^2 = 1 - \left(\pi^0 + \pi^1\right) = 1 - \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}}. \tag{8}$$

According to example 3.3, the probability of bid 2 being placed is $q^2 = \pi^2 + \pi^3 \approx 0.03078$. The approximation through the formula above leads to $q^2 = 1 - \sqrt[6]{\frac{5}{6}} \approx 0.02993$.

In general, a bid b is placed with probability

$$q^b = 1 - \sum_{k=0}^{b-1} \pi^{\nu_k} = \max\left(1 - \sqrt[N-1]{\sum_{k=0}^{b-1} \frac{c}{A-(k+1)}}; 0\right).$$

Figure 1 shows the shape of the probabilities of bids for varying number of participants in a LUPA with $A = 100$ and $c = 1$.

4 Data and Comparison

4.1 Data description

The data for seven German LUPAs was provided by *Legion Telekommunikation GmbH.* Data for other LUPAs is available online from *AuctionAir Ltd.*.

For the German auctions, the following data about each bid are known (see Table 3):

Bidder ID	**Channel**	**Time**	**Bid**
∼Tel.Nr.	SMS TEL WWW	DD.MM-HH.MM.SS.	€€€€.cc

Table 3. Registered Data

Bidders are identified by their telephone numbers. It is possible that one bidder uses several telephone numbers or that several bidders form a coalition and bid from different telephone numbers. In first case, the data would treat one player as two distinct bidders. In the second case, the data would identify one coalition as several distinct bidders. Bosch-Domenech et al. (2002) also note that in field experiments conducted via mass media coalition formation cannot be excluded. In some cases, one may suspect that coalitions were formed in a LUPA[3], but they seem to be extremely rare.

The bidding channel identifies how a bid was placed. Different LUPAs offer different possibilities for placing a bid. Most auctions we analyze here were based on SMS or telephone bidding. In some cases online bidding was also an option (then a registration was required to enable billing through a telephone company). LUPAs organized by AuctionAir™ offered only online bidding with credit card payment. In this case, bidders are identified by invoice numbers. One could place several bids with one invoice. If a bidder places bids through several invoices, however, then it is hard to identify these invoices with an individual player.

The German LUPAs provide us with exact time and date of the bids. AuctionAir™ only provides the date of the bid.

For the analysis in this paper we only use data about the bids. This information suffices to derive the frequency of each bid from the bids of all players. This frequency distribution is independent on the identification of bidders, hence missing information cannot influence the result.

[3] For example, two different players are suspected to build a coalition, if bidding from two different telephone numbers represents two complementing parts of one systemic strategy. Say, if bidder A places all bids from 1 to 1000, and bidder B places all bids from 1001 to 2000, we might suspect the two are in coalition. The winner of one LUPA admitted in an interview after the game that she played in a coalition with a friend.

4.2 Number of Bidders versus Number of Players

In order to evaluate bidding behavior in the LUPAs, we first need to estimate the number of participants in the game. From the data we know only the number of bidders N_{bid}, which is distinct from the number of players N since not to bid is a strategy. For a given number of players N , the number of bidders should average to

$$N_{bid} = \left(1 - \pi^0\right) N = \left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right) N, \tag{9}$$

with the rest of players choosing strategy s^0 of not participating in the bidding.

If we take the number of radio listeners as a proxy for the number of potential bidders, we compute for a radio station with 800 000 listeners and a LUPA with a prize $A = 10000€$ and bidding costs $c = 0,49€$ the number of 10 active bidders. The actual number of bidders, however, was close to 10000 persons!

On the other hand, if players do not follow the equilibrium structure derived in by Proposition 3.1, the representation (9) is not true. Hence, it may be more reasonable to estimate the number of potential bidders using the frequency of the first bid,

$$N = \frac{\ln \frac{c}{A-1}}{\ln\left(1 - q^1\right)} + 1. \tag{10}$$

This approach also hugely underestimation of the number of participants. For example, in a LUPA with the prize of $A = 10000€$ and bidding costs $c = 0,49€$, the number of bids $b = 1$ totalled 445 whereas the total number of bidders was 9400, and the total number of bids was 47872. If each of 9400 bidders places bid $b = 1$ with probability q^1, then the frequency of bid $b = 1$ should average to $9400q^1$, which allows us to estimate $q^1 \approx 0.04734$. As a result, the estimated number of players in the game would be $N = \frac{\ln \frac{49}{10^6}}{\ln(1-0.04734)} + 1 \approx 253$. This estimate of the total number of players is almost forty times below the number of bidders, and would only approach the actual number of bidders, if bid $b = 1$ were placed only by 12 bidders!

This suggests that we need another proxy for the number of players. In figures 2 - 5, we use numbers N_{bid}, $N_{bid} + 50\%$ and $N_{bid} - 50\%$ as possible proxies.

It is interesting to note that the LUPA in Fig. 5 followed the one shown in Fig. 4 with the same radio station just 3 days after the end of the latter. It seems that the experience of players, who were informed about the results of the first LUPA of the two in the series, changed the distribution of frequencies, bringing it closer to our theoretical approximation.

4.3 Frequencies of bids

Figure 6 compares the frequencies of bids in four LUPAs with equal costs (0,49€), values of

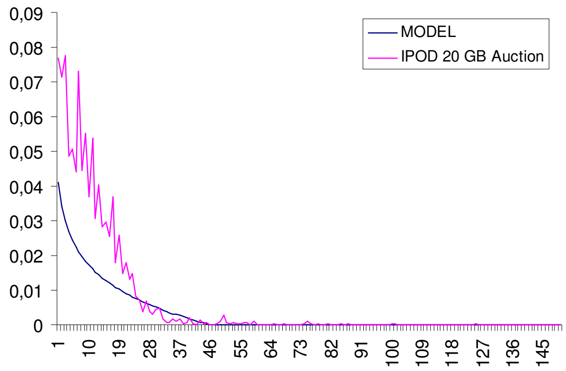

Figure 2. Comparison of the model predictions and actual bidding behavior in an Auction with 20Gb iPod as a prize: Number of participants 95 (known to players), approximate price of the lot 209 Pounds (known to players), bidding cost 4 pounds, averaged over 26 auctions.

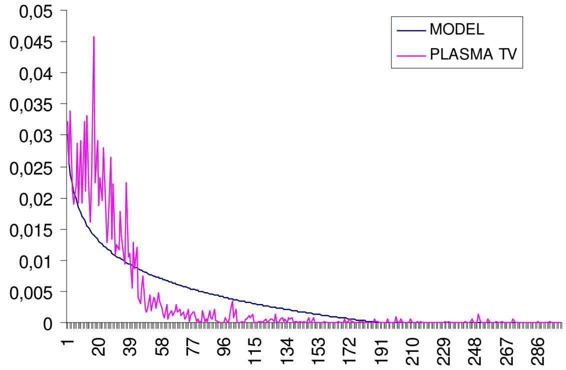

Figure 3. Comparison of the model predictions and actual bidding behavior in an Auction with a 42" Plasma TV from LG as a prize: Number of participants 175 (known to players), approximate price of the lot 2000 Pounds (known to players), bidding cost 10 pounds, averaged over 9 auctions.

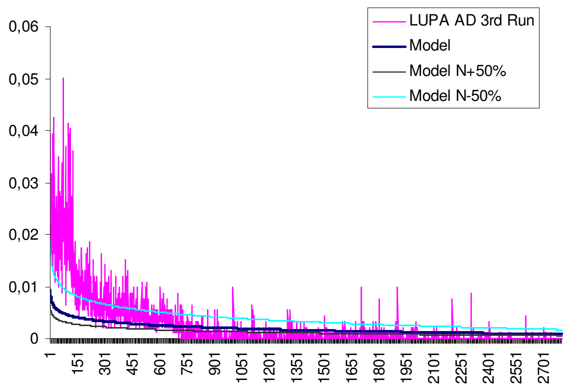

Figure 4. LUPA with Prize 3000€, bidding costs 0,49€, number of bidders 916, duration 4 days.

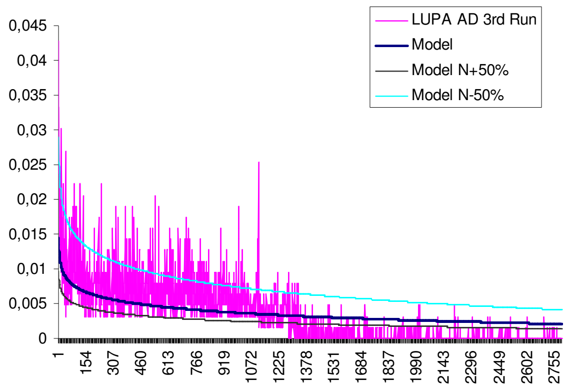

Figure 5. LUPA Antenne Düsseldorf, 631 bidders, Prize 5000€, bidding costs 0,49€, duration 5 days.

prizes ranging from 1000€ to 5000€, and numbers of bidders between 437 and 916. The figure reveals that the frequency distribution crucially depends on the value of the prize, but there is no significant difference in the frequencies between the LUPAs with monetary prizes and commodity prizes of equal values. Note that in the newspaper LUPA the lower limit for bids was set at the level of 1€, whereas in other auctions it was set on 0,01€. The auctions in Figure 6 were of different duration, but the differences in frequency distributions may not be attributed to the duration of the auction, as the following example shows.

In figure 7, two LUPAs with identical monetary prize of 10000€ and identical costs of0,49€ are compared. These two auctions were run by different radio stations in two different regions of Germany. The first auction ("RB" in the figure) had a bidding phase of 19 days, while the bidding phase in the second auction ("AMV" in the figure) was only 8 days long. The "RB"-auction took place 12 days after the end of the "AMV"-auction.

Although the bidding phase in the "AMV"-auction was shorter, it resulted in a slightly higher number of bidders. The duration of the bidding phase itself does not seem to have a significant influence the resulting frequencies of bids.

The same conclusion appears to hold for two LUPAs with significantly higher non-monetary stakes, as shown in Figure 8. Both plots show bids below 100€ and corresponding absolute frequencies in a range between 0 and 2000.

4.4 Strategies played in experiments

A symmetric mixed-strategy Nash equilibrium determines a probability distribution π^t over the pure strategies s^t. If the number of players who take part in the game is a large enough, then the share of players choosing strategy s^t will be approximately equal to this probability π^t.

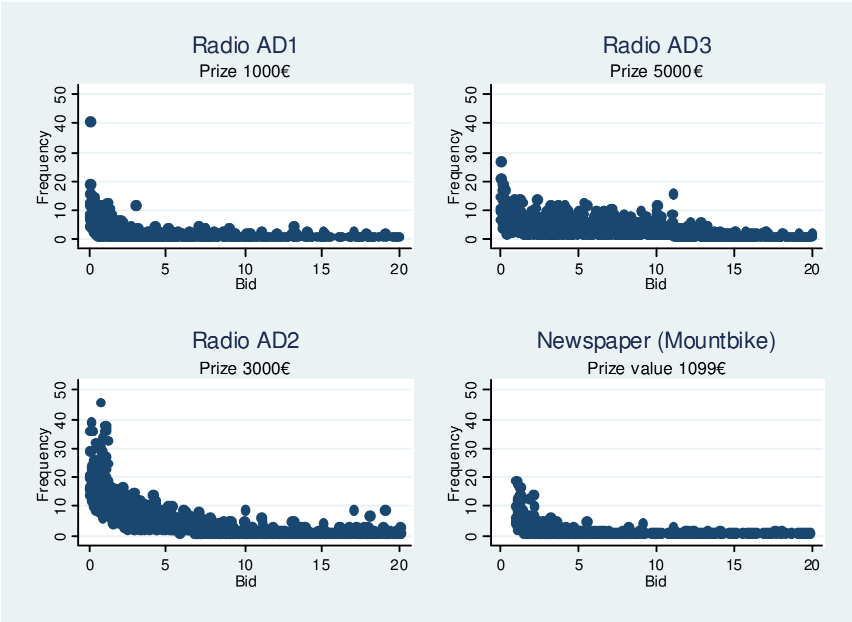

Figure 6. Frequencies of bids in four LUPAs with comparable prizes, costs and number of participants.

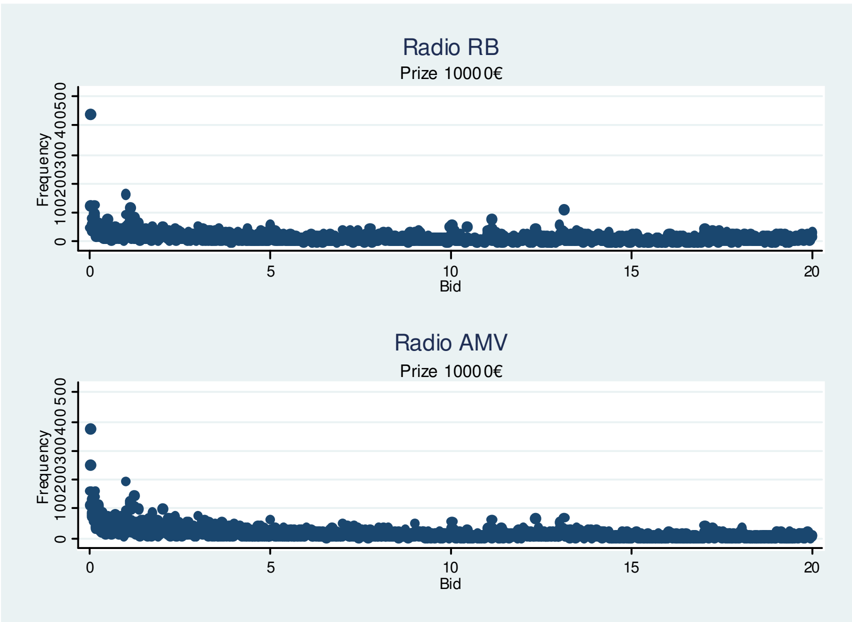

Figure 7. Frequencies of bids in two identical LUPAs with different duration of the bidding phase.

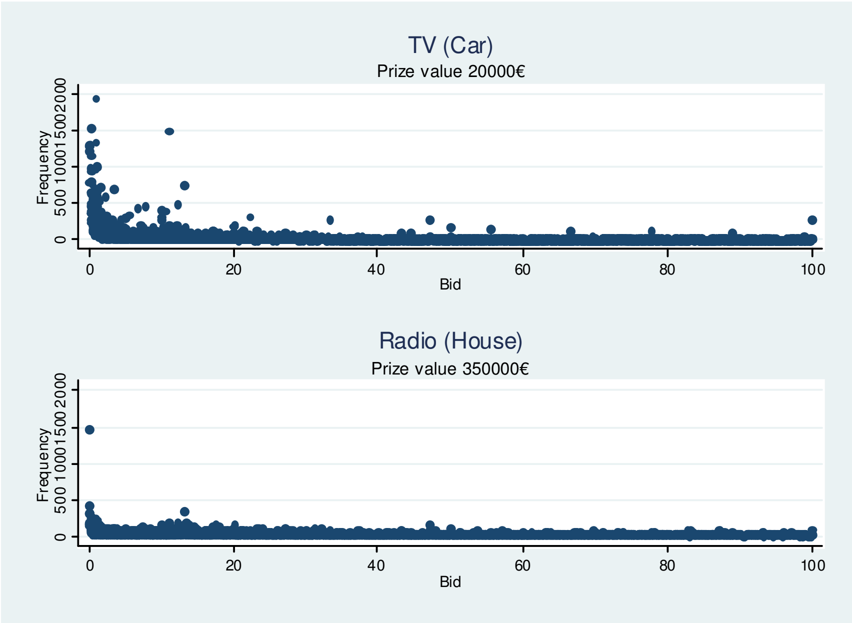

Figure 8. Frequencies of bids in LUPAs with extremely different big stakes and different duration of the bidding phase.

For the approximate equilibrium of Proposition 3.1 the equilibrium strategies which are played with positive probabilities have the property that placing a bid k implies also placing the lower bid $k-1$. Accordingly, equilibrium bids should be systematically cover all bids up to a certain level. If all players were to bid in this way, the frequency distribution over bids must be a decreasing function.

Figure 9 shows the frequency distribution of bids in the interval from €10 to €20 in a LUPA with about 10000 active participants and a the prize of €10000. The data reveals that bidders do not seem to restrict their behavior to "systematic" bidding. For example, the share of players, who place a bid on €11.11 is about 100 times higher then the share of players who bid on €10.98. Similarly, the frequency of bids on €13.13 exceeds the frequency on €12.70.

Figure 10 shows the bids (horizontal axis) made by individual bidders (vertical axis) in a LUPA with a mountain bike worth 1099€ as a prize in which 437 bidders took part. If players would follow systemic strategies, the diagram should show solid horizontal bars. Figure 11 shows an enlargeemnt of the diagramp for bids from 1€ to 15€.; The circle on the vertical axis indicates a bidder (with identification number 372), who placed a total of 83 bids closely covering the interval between 1€ and 2€.

Figure 9 reveals an interesting property of the frequency curve. A close-up of the frequency distribution reveals spikes following a regular pattern. Figure 12 shows for two LUPAs three

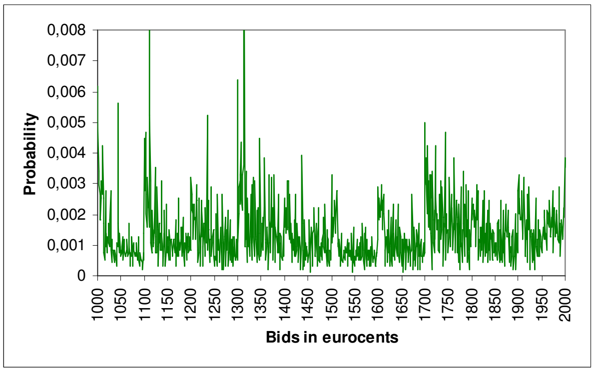

Figure 9. LUPA with prize 1000€, number of bidders 9400, bidding costs 0,49€, duration 17 days. Only bids 10,00€ - 20,00€ are shown. The winning bid is 14,55€.

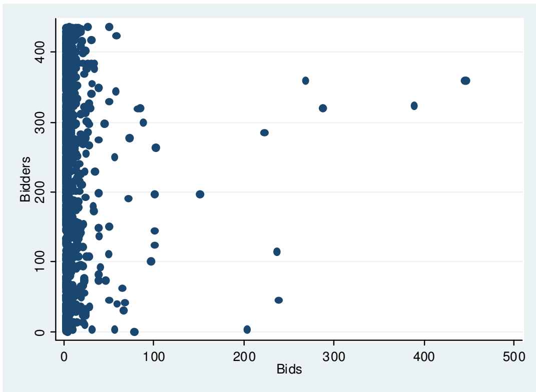

Figure 10. Strategic space: each bidder is uniquely identified by a point on the vertical axis with dots in the plot identifying bids placed by this bidder.

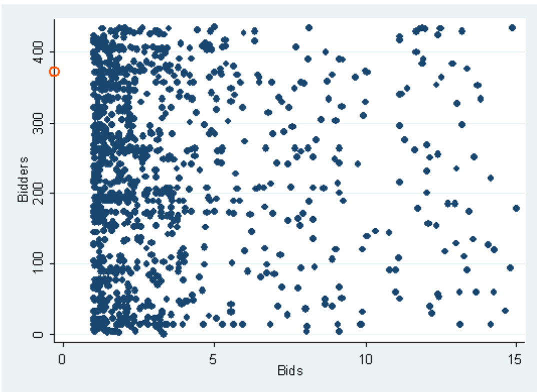

Figure 11. Strategic space: bidders (vertical axis) and their bids (horizontal axis). A circle on the vertical axis points at a systemic bidder.

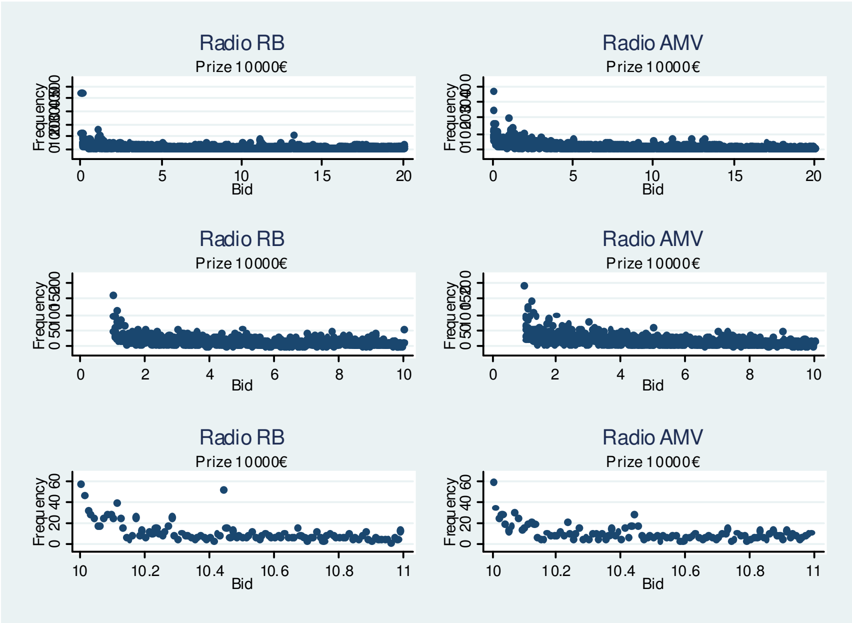

Figure 12. "Close up" of the frequency curve from interval (0;20) to (1;10) and to (10;11) for two similar LUPAs.

close-ups of the frequency curve. The shape of the curve is about the same no matter whether it is considered at the interval from 0 to 20€, from 1€ to 10€ or from 10€ to 11€. This fractal property of the frequency curve is difficult to explain either by the game-theoretic model or by random bidding.

A possible explanation of the spikes and of the fractal structure may be found in heterogeneity of agents. Figure 13 illustrates this idea. Suppose a group of players would bid for some reason only above 1€, then we would observe a spike at the bid 100 (cents). A possible justification for not bidding below 1€ bids may be a conjecture that the number of rivals bidding below 1€ is high enough to reduce the probability of winning with a bid below 100 cents to zero. Such reasoning may lead players to "eliminate bids which are probably dominated". One might call these players "super-rational", although this is incompatible with an equilibrium strategy.

It remains, however, unexplained why other "super rational" players would choose to take the numbers 100, 200, 300, .. 1000 etc. as the lowest bound for their "reduced set of bids". Even a theory of prominent numbers (Albers 1997) would not support this set of "prominent numbers" since they cannot be reduced to the base of 1, 2, or 5.

5 Discussion

Least unmatched price auctions provide a field experiment for testing Nash equilibria in

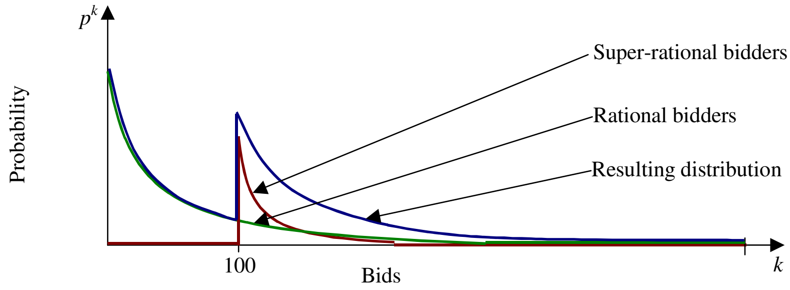

Figure 13. Possible explanation for spikes in the frequency distribution: "super-rational" bidders first reduce the bidding interval, e.g. by making a guess that no bid below 1€ can win. After that they bid as though the game would only allow bids higher than 1€.

mixed strategies. We have derived an approximation to a symmetric Nash equilibrium of these LUPAs. Our model explains the downward slope of the bid distribution observed in the data, and its dependence on the prize and bidding cost. However, the actual curvature of the bid distribution deviates from the one predicted by the model. Partially this may be due to the approximation used for the derivation of the symmetric Nash equilibrium in the game-theoretic model.As Figures 4 and 5 suggest, however, that experience leads to a curvature which is closer to the theoretical prediction. Moreover, a comparison of different LUPAs with monetary and non-monetary payoffs shows no significant difference in the aggregate behavior of players.

The data also reveals some other features, which may be hard to reconcile with any game-theoretic model. For example, it seems unlikely that the fractal structure of the bid frequencies can find an explanation in the bid distribution of a Nash equilibrium. These features require additional research both into the unconstrained bid distribution of a symmetric mixed-strategy Nash equilibrium and into alternative behavioral assumptions about the participants of these auctions.

Appendix A. Proofs

Proof of Proposition 2.2

I. $c > A - 1$ implies that $p_i(s) < 0$, $\forall i \in I$, $\forall s_1, .., s_{i-1}, s_{i+1}, ..s_m \in S$, $\forall s_i \in S : s_i \neq \varnothing$. Obviously, $s_1^* = .. = s_m^* = \varnothing$ is a Nash Equilibrium in Pure Strategies (NEPS) with $p_i(s^*) = 0$, $\forall i \in I$. To prove uniqueness, assume s' is an equilibrium combination of strategies, where at least one player i plays $s_i \neq s^0 = \varnothing$. Then for this player $p_i(s') < 0 = p_i(s^*)$ where $s^* : s_1^* = .. = s_m^* = \varnothing$. Hence, s' is not a NEPS.

II. $c = A - 1$ implies there are no strategy combinations with strictly positive payoffs: for any player i any strategy s_i with $|s_i| > 1$ leads to $p_i(s) < 0$ for any combination of other players' strategies s_{-i}, and are hence dominated by the trivial strategy, therefore they cannot be NEPS. The strategies $s_i = \{b\}$ (consisting only of one bid, $|s_i| = 1$) lead to strictly negative payoffs, as soon as $b > 1$, for any combination of other players' strategies s_{-i}, and are also dominated by the trivial strategy, therefore they cannot be NEPS. The set of strategies, which can lead to an equilibrium is therefore reduced to $S = \{1\} \bigcup \varnothing$. It is easy to check that a combination $s^* : s_1^* = .. = s_m^* = \varnothing$ is a NEPS. Any combination $s^{**} = \left(s_i^{**}, s_{-i}^{**}\right) = (\{1\}, \varnothing)$, $\forall i \in I$ is a NEPS since $\beta(s^{**}) = 1$ and hence $p_i(s^{**}) = 0 = p_i\left(\varnothing, s_{-i}^{**}\right)$. The number of NEPS is $m+1$.

III. $c < A-1$ implies there exist strategy combinations with strictly positive payoff of one player. For example, a strategy combination $s = (s_i, s_{-i}) = (\{1\}, \varnothing)$, $\forall i \in I$ leads to a positive payoff of the player i: $p_i(s) = A - 1 - c > 0$. Therefore the strategy combination $s^* : s_1^* = .. = s_m^* = \varnothing$ is not a NEPS anymore (each player has incentives to deviate in favour of a strategy with a positive payoff). The combination $\left(s_i^*, s_{-i}^*\right) = (\{1\}, \varnothing)$ is a NEPS since $p_i\left(s_i^*, s_{-i}^*\right) = p_i(\{1\}, \varnothing) > 0 = p_i(\varnothing, \varnothing)$, and $p_{-i}\left(s_{-i}^*, s_i^*\right) = p_{-i}(\varnothing, \{1\}) = 0 \geq p_{-i}(s_{-i}, \{1\})$ for any other strategy $s_{-i} \in S$. We have equality only if a player j plays $s_j' = \{1; 2\}$.

Any strategy combination s', where at least two players play $\{1\}$ and all other players play the trivial strategy, cannot be a NEPS since the set of unmatched bids is empty and the payoffs of the players, who play $\{1\}$, is negative, which creates incentive for them to deviate in favour of the trivial strategy.

Any strategy combination s'', which contains strategies different from $\{1\}$ and $\varnothing$, cannot be a NEPS since either such strategies contain a bid $b \geq 2$, or their length is strictly greater than one, which both induce negative payoff and hence create incentives to deviate in favour of the trivial strategy.

This proves that the strategy combinations $\left(s_i^*, s_{-i}^*\right) = (\{1\}, \varnothing)$ are the only NEPS. The number of such combinations is m.

IV. $c \gtreqless \frac{A}{2} - 1$. First, note that $s_1^* = .. = s_m^* = \varnothing$ is not a NEPS since each player has incentives to deviate in favour of a strategy with positive payoff. A strategy combination without winner cannot be a NEPS (as above).

Assume $s^* = (s_1^*, ..s_m^*)$ is a NEPS, and the player i wins. This implies that the strategies played by all other players are trivial: $s_j^* = \varnothing$, $\forall j \in I : j \neq i$ (if this would not be the case, it would imply that some players who loose, obtain negative payoff and hence have incentive to deviate in favour of the trivial strategy, which contradicts to the assumption that s^* is a NEPS). This implies that $s_i^* = \{1\}$, as this is the strategy with maximum payoff (otherwise the player

i has incentive to deviate in favour of $\{1\}$, which would contradict to the assumption that s^* is a NEPS). This implies that if a player $j \neq i$ plays a strategy $s'_j = \{1,2\}$, he can become a winner with the winning bid $b = 2$. This deviation is only profitable compared to the initial strategy $s^*_j = \varnothing$ if $p_j = A - 2 - 2c > 0$. As soon as $c \geq \frac{A}{2} - 1$, no player j would have incentives to deviate from the trivial strategy, and hence a strategy combination $\left(s^*_i, s^*_{-i}\right) = (\{1\}, \varnothing)$ is a NEPS. Otherwise any player $j \neq i$ has an incentive to deviate from the strategy $s^*_j = \varnothing$ in favour of the strategy $s'_j = \{1,2\}$. This is a contradiction to the assumption that s^* is a NEPS. Therefore no NEPS can exist if $c < \frac{A}{2} - 1$. ■

Proof of Lemma 2.3

Consider expected payoff of player i (2). With respect to expected costs, we can write:

$$\begin{aligned}
c \sum_{s \in S^N} \pi(s) \sum_{k=1}^{\mathbf{b}} s_i(k) &= c \sum_{s \in S^N} \left(\prod_{i=1}^{N} \pi_i(s_i) \right) \sum_{k=1}^{\mathbf{b}} s_i(k) = \\
&= c \sum_{s \in S^N} \left(\prod_{j \neq i} \pi_j(s_j) \right) \pi_i(s_i) \sum_{k=1}^{\mathbf{b}} s_i(k) = \\
&= c \sum_{s_i \in S} \sum_{s_{-i} \in S^{N-1}} \sum_{k=1}^{\mathbf{b}} \pi_i(s_i)\, s_i(k) \left(\prod_{j \neq i} \pi_j(s_j) \right) = \\
&= c \sum_{s_i \in S} \sum_{k=1}^{\mathbf{b}} \pi_i(s_i)\, s_i(k) \sum_{s_{-i} \in S^{N-1}} \left(\prod_{j \neq i} \pi_j(s_j) \right) = \\
&= c \sum_{s_i \in S} \sum_{k=1}^{\mathbf{b}} \pi_i(s_i)\, s_i(k)
\end{aligned}$$

and hence

$$c \sum_{s \in S^N} \pi(s) \sum_{k=1}^{\mathbf{b}} s_i(k) = c \sum_{t=0}^{2^{\mathbf{b}}-1} \sum_{k=1}^{\mathbf{b}} \pi_i^t s^t(k) \tag{A-1}$$

since $\sum_{s_{-i} \in S^{N-1}} \left(\prod_{j \neq i} \pi_j(s_j) \right)$ does not depend on s_i and equals to unity, for it describes the probability of reaching any of strategy combinations in a subgame determined by a given strategy s_i of player i.

Now we have to determine the expected prize of player i:

$$\begin{aligned}
\sum_{s \in S^N} \pi^s (A - \mu(s))\, s_i(\mu(s)) &= \sum_{s \in S^N} s_i(\mu(s)) \prod_{j=1}^{N} \pi_j(s_j)\,(A - \mu(s)) = \\
&= \sum_{k=1}^{\mathbf{b}} \sum_{\substack{s \in S^N: \\ \mu(s) = k}} s_i(k) \prod_{j=1}^{N} \pi_j(s_j)\,(A - k)
\end{aligned} \tag{A-2}$$

The latter may be written in a form

$$\sum_{k=1}^{\mathbf{b}} (A-k) \sum_{s \in S^N} s_i(k) \prod_{j=1}^{N} \pi_j(s_j) \rho(s,k) \psi(s,k)$$

Here $\rho(s,k)$ and $\psi(s,k)$ are boolean functions:

$$\rho(s,k) = \begin{cases} 1 & \text{if } \sum_{j=1}^{N} s_j(k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi(s,k) = \begin{cases} 1 & \text{if } \sum_{j=1}^{N} s_j(l) \neq 1, \forall l < k \\ 0 & \text{otherwise} \end{cases}$$

We only need to determine functions $\rho(s,k)$ and $\psi(s,k)$. It is easy to see that $\sum_{j=1}^{N} s_j(k) = 1$ if and only if $\sum_{j=1}^{N} s_j(k) \prod_{h \neq j} (1 - s_h(k)) = 1$, therefore we can choose the latter to be $\rho(s,k)$. With regards to $\psi(s,k)$, note that for any strategy combination s and for any bid l, the above function $\rho(s,l)$ only delivers unity if $\sum_{j=1}^{N} s_j(l) = 1$, and is zero otherwise. The term $\prod_{l=1}^{k-1} (1 - \rho(s,l))$ is then unity only if no $l < k$ is placed exactly once in a given strategy combination s. Therefore, we can choose $\psi(s,k)$ to be

$$\psi(s,k) = \prod_{l=1}^{k-1} \left[1 - \sum_{j=1}^{N} s_j(l) \prod_{h \neq j} (1 - s_h(l)) \right]$$

■

Proof of Proposition 3.1

To prove that π^* is an equilibrium, assume that each rival of player i plays the mixed strategy from π^*. Starting from this point, we show that if player i also plays the same mixed strategy, he has no incentives to deviate from it.

First, consider the probability of playing s^1. Player i chooses π_i^1 to maximize his expected payoff $P_i(\pi_i, \pi_{-i})$. The marginal payoff is independent of $\pi_i^2..\pi_i^{2^{\mathbf{b}}-1}$, due to the linearity of the payoff function. Since other players only play s^{ν_k} with positive probabilities, there is only one strategy combination s_{-i} played with positive probability, in which $s_i = s^1$ wins A, namely if all other players play s^0. In all other strategy combinations either some other player plays a strategy, which includes bid "1", and hence strategy $s_i = s^1$ cannot win, or the strategy combination includes strategies, which are played with zero probability, and hence the whole strategy combination is played with zero probability. The combination of mixed strategies π_{-i}^* of $N-1$ rivals of i implies that the only strategy combination s_{-i}, for which $\rho\left(\left(s^1, s_{-i}\right), 1\right) \cdot$

$\psi\left(\left(s^1, s_{-i}\right), 1\right) = 1$, is $s_{-i} = \left(s^0..s^0\right)$ and it is played with probability $\left(\pi^0\right)^{N-1}$. The choice of mixed strategies (6) implies

$$\pi^0 = \sqrt[N-1]{\frac{c}{A-1}}$$

which makes the marginal expected profit for player i equal to zero:

$$\frac{\partial P_i\left(\pi_i, \pi_{-i}\right)}{\partial \pi_i^1} = (A-1)\left(\pi^0\right)^{N-1} - c = 0 \tag{A-3}$$

The first order condition A-3 is derived from the payoff function of player i using the fact, that the boolean functions ψ and ρ are only unity for the strategy combination, in which all other players play s^0. Condition A-3 implies that player i is indifferent with regards to probability π_i^1 of playing s^1. In other words, player i cannot be better off if he deviates from playing $\pi_i^1 = \pi^*\left(s^1\right)$.

In a similar way, FOC for $\pi_i^{\nu_k}$ is met if strategies $s^{\nu_0}..s^{\nu_{k-1}}$ are played with probabilities given by π^*. For each bid $l = 1..k$ in strategy s^{ν_k}, we need to determine strategy combinations s_{-i}, for which $\rho\left(\left(s^{\nu_k}, s_{-i}\right), l\right) \cdot \psi\left(\left(s^{\nu_k}, s_{-i}\right), l\right) = 1$. These are strategy combinations, in which player i would win with bid l. All other strategy combinations enter the expected payoff function with zero weight.

Let us denote with S_{-i}^l the set of strategy combinations s_{-i} of the rivals of i, in which player i can win with any bid $m = 1..l$. Since all $N-1$ rivals of i play mixed strategy combination π_{-i}^*, we can calculate the cumulative probability of all strategy combinations, in which player i can win with any bid $m = 1..l$: $\Pr\left(S_{-i}^l\right) = \left(\sum_{m=0}^{l-1} \pi^{\nu_m}\right)^{N-1}$.[4] Among these strategy combinations we are only interested in those combinations, in which only bid $m = l$ wins. In terms of sets of strategy combinations, we are looking for strategy combinations $s_{-i} \in S_{-i}^l \backslash S_{-i}^{l-1}$. To determine the probability of such strategy combinations, we need to subtract from $\Pr\left(S_{-i}^l\right)$ the probability $\Pr\left(S_{-i}^{l-1}\right) = \left(\sum_{m=0}^{l-2} \pi^{\nu_m}\right)^{N-1}$ of player's i rivals' playing strategy combinations, in which player i can win with some lower bid $m = 1..l-1$. Therefore, for any

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[4] For example, if players $-i$ play $\left(s^0..s^0\right)$, then player i wins with bid 1. In this notation $s^{\nu_0} = s^0$. If players $-i$ play $\left(s^{\nu_l}, s^0..s^0\right)$, then player i wins with bid $l+1$ (of course, if this bid is in his strategy, i.e. if $s_i(l+1) = 1$). Whilst strategy combination $\left(s^0..s^0\right)$ should be counted only once, the combination $\left(s^{\nu_l}, s^0..s^0\right)$ should be counted $N-1$ times, since *any* of $N-1$ players can choose to play s^{ν_k}, whereas other players play s^0. Hence, we need to count for *all* possible ways, in which $N-1$ players can choose among l distinct strategies $s^{\nu_0}..s^{\nu_l}$ so that $n_0$of players play s^{ν_0}, $n_1$of players play s^{ν_1}, and so on. Obviously, $\sum_{h=1}^{l} n_h = N-1$ (the sum of numbers of players who play strategy s^{ν_h} over all strategies is equal to the total number of players under consideration). The number of combinations of each type is given by multinomial coefficients $C_{n_0,n_1..n_l}^{N-1} = \begin{pmatrix} N-1 \\ n_0, n_1..n_l \end{pmatrix} = \frac{(N-1)!}{n_0! n_1! \cdots n_l!}$. For each strategy combination, its probability is given by $\prod_{j \neq i} \pi_j = \left(\pi^{\nu_0}\right)^{n_0}\left(\pi^{\nu_1}\right)^{n_1} \cdots \left(\pi^{\nu_l}\right)^{n_l}$. Using the multinomial theorem, we can write down the sum of probabilities of all such strategy combinations as $\left(\sum_{m=0}^{l} \pi^{\nu_m}\right)^{N-1}$.

bid l from given strategy s^{ν_k} played by player i, we may write

$$\sum_{s_{-i}\in S_{-i}} \pi(s_{-i})\,\rho((s^{\nu_k},s_{-i}),l)\,\psi((s^{\nu_k},s_{-i}),l) = \left(\sum_{m=0}^{l-1}\pi^{\nu_m}\right)^{N-1} - \left(\sum_{m=0}^{l-2}\pi^{\nu_m}\right)^{N-1}$$

The discussion above guarantees that the right-hand side counts for those strategy combinations, in which $\rho((s_i,s_{-i}),l)\,\psi((s_i,s_{-i}),l)=1$. We do not need to count for the rest of them, since $\rho((s_i,s_{-i}),l)\,\psi((s_i,s_{-i}),l)$ is a boolean function and it equals zero if not equal unity.

The marginal expected payoff of player i from playing strategy s^{ν_k} with probability $\pi_i^{\nu_k}$ is hence given by

$$\frac{\partial P_i(\pi_i,\pi_{-i})}{\partial \pi_i^{\nu_k}} = \sum_{l=1}^{k}(A-l)\left[\left(\sum_{m=0}^{l-1}\pi^{\nu_m}\right)^{N-1} - \left(\sum_{m=0}^{l-2}\pi^{\nu_m}\right)^{N-1}\right] - kc \tag{A-4}$$

Substituting for π^{ν_m} from π^*_{-i} yields

$$\frac{\partial P_i(\pi_i,\pi_{-i})}{\partial \pi_i^{\nu_k}} = 0$$

Again, player i is indifferent with regards to $\pi_i^{\nu_k}$, and deviating from $\pi_i^{\nu_k} = \pi^*(s^{\nu_k})$ does not make player i better off.

Choosing $\pi_i^{\nu_k}$ iteratively, player i approaches strategy s^{ν_M} such that setting $\pi_i^{\nu_M}$ in accordance with the formula for $\pi_i^{\nu_k}$ would lead to $\sum_{m=0}^{M}\pi_i^{\nu_m} > 1$. The first order condition A-4 for $\pi_i^{\nu_M}$ is met by the choice of π^*_{-i} for strategies $s^{\nu_0}..s^{\nu_{M-1}}$, hence if player i plays $\pi_i^{\nu_M}$ with smaller probability, which makes $\sum_{m=0}^{M}\pi_i^{\nu_m}=1$, he is still in his optimum.

■

Solution to Example 3.1

Consider players i and j playing a two-players LUPA with the prize of 4 and marginal costs of 1. Obviously, strategies with bids $b>3$ are strictly dominated by the non-entrance strategy. Expected payoff function is in this case

$$\begin{aligned} P_i(\pi_i,\pi_{-i}) \;=\;& 3\sum_{s\in S^N}\pi_i(s_i)\,s_i(1)\pi_j(s_j)\,\rho(s,k)\,\psi(s,k) + \\ & 2\sum_{s\in S^N}\pi_i(s_i)\,s_i(2)\pi_j(s_j)\,\rho(s,k)\,\psi(s,k) + \\ & \sum_{s\in S^N}\pi_i(s_i)\,s_i(3)\pi_j(s_j)\,\rho(s,k)\,\psi(s,k) - c\sum_{t=0}^{7}\sum_{k=1}^{3}\pi_i^t s^t(k) \end{aligned}$$

The strategy set of each player consists of eight strategies $s^0..s^7$, following the notation from the beginning of the paper. There are in total $8^2=64$ strategy combinations.

The first summation operator only includes those strategy combinations, in which player

i places bid 1, otherwise $s_i(1) = 0$. These are all strategy combinations within which player i plays strategies $s^1 = (1,0,0)\,, s^3 = (1,1,0)$, $s^5 = (1,0,1)\,,\ s^7 = (1,1,1)$. The boolean function $\rho(s,1) = s_i(1)(1-s_j(1)) + s_j(1)(1-s_i(1)) = 1 - s_j(1)$ only delivers unity if player j plays strategies with even numbers: $s^0 = (0,0,0)\,, s^2 = (0,1,0)$, $s^4 = (0,0,1)\,,\ s^6 = (0,1,1)$. The boolean function $\psi(s,1)$ delivers unity, since there are no bids below $b = 1$. The first summation operator transforms into

$$3\sum_{t=0}^{3}\pi_i^{2t+1}\sum_{u=0}^{3}\pi_j^{2u}$$

The second summation operator only includes strategy combinations, in which player i places bid $b = 2$, i.e. those with player's i strategies $s^2 = (0,1,0)$, $s^3 = (1,1,0)$, $s^6 = (0,1,1)\,$, $s^7 = (1,1,1)$. The boolean function $\rho(s,2) = 1 - s_j(2)$ only delivers unity for following strategies played by player j: $s^0 = (0,0,0)\,, s^1 = (1,0,0)$, $s^4 = (0,0,1)\,,\ s^5 = (1,0,1)$. The boolean function $\psi(s,2) = 1 - s_i(1)(1-s_j(1)) - s_j(1)(1-s_i(1))$ only delivers unity for strategy combinations (s^2,s^0), (s^2,s^4), (s^3,s^1), (s^3,s^5), (s^6,s^0), (s^6,s^4), (s^7,s^1), (s^7,s^5) from those defined above. Therefore, the second summation operator transforms into

$$\begin{aligned} 2\left(\left(\pi_i^2+\pi_i^6\right)\left(\pi_j^0+\pi_j^4\right)+\left(\pi_i^3+\pi_i^7\right)\left(\pi_j^1+\pi_j^5\right)\right) &= \\ &= 2\sum_{t=0}^{1}\left(\pi_i^{2+t}+\pi_i^{6+t}\right)\left(\pi_j^{t}+\pi_j^{4+t}\right) \end{aligned}$$

The third summation operator only includes strategy combinations, consisting of $s^4 = (0,0,1)$, $s^5 = (1,0,1)$, $s^6 = (0,1,1)$, $s^7 = (1,1,1)$ for player i, and of $s^0 = (0,0,0)$, $s^1 = (1,0,0)$, $s^2 = (0,1,0)$, $s^3 = (1,1,0)$ for player j. Among these strategy combinations, function

$$\begin{aligned} \psi(s,3) &= \left(1 - s_i(1)(1-s_j(1)) - s_j(1)(1-s_i(1))\right)\times \\ &\quad \left(1 - s_i(2)(1-s_j(2)) - s_j(2)(1-s_i(2))\right) \end{aligned}$$

only delivers unity for (s^4,s^0), (s^5,s^1), (s^6,s^2), (s^7,s^3). Therefore, the third summa transforms into

$$\sum_{t=0}^{3}\pi_i^{t+4}\pi_j^{t}$$

The cost term for player i takes the form

$$\pi_i^1+\pi_i^2+\pi_i^4+2\left(\pi_i^3+\pi_i^5+\pi_i^6\right)+3\pi_i^7$$

Summarizing, the expected payoff function for player i may be written as

$$\begin{aligned} P_i(\pi_i,\pi_j) &= 3\sum_{t=0}^{3}\pi_i^{2t+1}\sum_{u=0}^{3}\pi_j^{2u}+2\sum_{t=0}^{1}\left(\pi_i^{2+t}+\pi_i^{6+t}\right)\left(\pi_j^{t}+\pi_j^{4+t}\right)+\sum_{t=0}^{3}\pi_i^{t+4}\pi_j^{t} - \\ &\quad \left(\pi_i^1+\pi_i^2+\pi_i^4+2\left(\pi_i^3+\pi_i^5+\pi_i^6\right)+3\pi_i^7\right) \end{aligned}$$

Rearranging leads to the following payoff matrix (only payoffs for player i are shown, symmetric for j):

$i \setminus j$	s^0	s^1	s^2	s^3	s^4	s^5	s^6	s^7
s^0	0	0	0	0	0	0	0	0
s^1	2	−1	2	−1	2	−1	2	−1
s^2	1	−1	−1	−1	1	−1	−1	−1
s^3	1	0	1	−2	1	0	1	−2
s^4	0	−1	−1	−1	−1	−1	−1	−1
s^5	1	−1	1	−2	1	−2	1	−2
s^6	0	−2	−1	−2	0	−2	−2	−2
s^7	0	−1	0	−2	0	−1	0	−3

To proceed with the best response function of player i, we find following derivatives of his expected payoff function:

$$\begin{array}{ll}
\frac{\partial P_i}{\partial \pi_i^0} = 0 & \frac{\partial P_i}{\partial \pi_i^4} = \pi_j^0 - 1 \\
\frac{\partial P_i}{\partial \pi_i^1} = 3\sum\limits_{u=0}^{3} \pi_j^{2u} - 1 & \frac{\partial P_i}{\partial \pi_i^5} = 3\sum\limits_{u=0}^{3} \pi_j^{2u} + \pi_j^1 - 2 \\
\frac{\partial P_i}{\partial \pi_i^2} = 2\left(\pi_j^0 + \pi_j^4\right) - 1 & \frac{\partial P_i}{\partial \pi_i^6} = 2\left(\pi_j^0 + \pi_j^4\right) + \pi_j^2 - 2 \\
\frac{\partial P_i}{\partial \pi_i^3} = 3\sum\limits_{u=0}^{3} \pi_j^{2u} + 2\left(\pi_j^1 + \pi_j^5\right) - 2 & \frac{\partial P_i}{\partial \pi_i^7} = 3\sum\limits_{u=0}^{3} \pi_j^{2u} + 2\left(\pi_j^1 + \pi_j^5\right) + \pi_j^3 - 3
\end{array} \tag{A-5}$$

The probability of playing strategy s^0 results from the constraint $\sum\limits_{t=0}^{7} \pi_i^t = 1$.

The derivatives of the payoff function exhibit following structure:

$$\frac{\partial P_i}{\partial \pi_i^{t+4}} = \frac{\partial P_i}{\partial \pi_i^t} + \pi_j^t - 1, \forall t = 0..3 \tag{A-6}$$

It immediately follows that in an equilibrium, in which player j plays the first four strategies with $\pi_j^t < 1$, $t = 1..4$, player i plays $\pi_i^4 = \pi_i^5 = \pi_i^6 = \pi_i^7 = 0$. Indeed, either $\frac{\partial P_i}{\partial \pi_i^t} > 0$ and $\pi_i^t = 1$, which implies all other probabilities to be zero, or $\frac{\partial P_i}{\partial \pi_i^t} \leq 0$, which implies $\frac{\partial P_i}{\partial \pi_i^{t+4}} < 0$ and hence $\pi_i^{t+4} = 0$.

If player j plays $\pi_j^0 = 1$, then it is optimal for player i to play $\pi_i^1 = 1$. If player j plays $\pi_j^1 = 1$, all derivatives in (A-5) turn negative, except $\frac{\partial P_i}{\partial \pi_i^3} = 0$. Therefore, it is still optimal for player i to play $\pi_i^0 = 0$. At the same time, player i is indifferent between strategies s^0 and s^3, since both respective marginal payoffs are zero. Player i can choose $\pi_i^3 > 0$, respecting the constraint $\pi_i^0 + \pi_i^3 = 1$, but π_i^0 cannot be smaller than $\frac{1}{3}$, since otherwise $\pi_j^1 = 1$ is not optimal for player j anymore. This reasoning leads to two connected continua of equilibria

$$\begin{array}{lll}
\text{NE1} & \pi_i = (0,1,0,0,0,0,0,0) & \pi_j = (\alpha,0,0,1-\alpha,0,0,0,0) \\
\text{NE2} & \pi_i = (\alpha,0,0,1-\alpha,0,0,0,0) & \pi_j = (0,1,0,0,0,0,0,0)
\end{array}$$

with $\alpha \in \left[\frac{1}{3}, 1\right]$

There are no other equilibria with some player playing a pure strategy. For equilibria in mixed strategies we may focus on strategies $s^0..s^3$ (due to property A-6). Substituting for

$\pi_j^4 = \pi_j^5 = \pi_j^6 = \pi_j^7 = 0$ and for $\pi_j^0 = 1 - \pi_j^1 - \pi_j^2 - \pi_j^3$ in (A-5) yields

$$\frac{\partial P_i}{\partial \pi_i^1} = 2 - 3\pi_j^1 - 3\pi_j^3 \leq 0$$
$$\frac{\partial P_i}{\partial \pi_i^2} = 1 - 2\pi_j^1 - 2\pi_j^2 - 2\pi_j^3 \leq 0$$
$$\frac{\partial P_i}{\partial \pi_i^3} = \frac{\partial P_i}{\partial \pi_i^1} + 2\pi_j^1 - 1 \leq 0$$

The last condition requires that $\frac{\partial P_i}{\partial \pi_i^1} \leq 1 - 2\pi_j^1$. If $\pi_j^1 > \frac{1}{2}$, which is only possible if $\frac{\partial P_j}{\partial \pi_j^1} = 0$, we obtain $\pi_i^1 = 0$, which implies $\frac{\partial P_j}{\partial \pi_j^3} < 0$, and hence $\pi_j^3 = 0$. In this case in order to meet the first condition, π_j^1 should satisfy $2 - 3\pi_j^1 \leq 1 - 2\pi_j^1$, which is equivalent to $\pi_j^1 \geq 1$. Since pure strategies are not possible, the latter is a contradiction.

If $\pi_j^1 < \frac{1}{2}$, then $\frac{\partial P_i}{\partial \pi_i^3} < 0$ and $\pi_i^3 = 0$, therefore $\frac{\partial P_j}{\partial \pi_j^1} = 2 - 3\pi_i^1 \leq 0$ implies $\pi_i^1 \geq \frac{2}{3}$, which is only possible if $\frac{\partial P_i}{\partial \pi_i^1} = 0$. The latter requires $\frac{2}{3} - \pi_j^3 \leq \pi_j^1 < \frac{1}{2}$ and hence $\pi_j^3 > \frac{1}{6}$, which needs $\frac{\partial P_j}{\partial \pi_j^3} = \frac{\partial P_j}{\partial \pi_j^1} + 2\pi_i^1 - 1 = 0$ implying $\frac{\partial P_j}{\partial \pi_j^1} < 0$, and hence $\pi_j^1 = 0$. This implies $\frac{2}{3} - \pi_j^3 \leq \pi_j^1 = 0$ i.e. $\pi_j^3 \geq \frac{2}{3}$ and in turn $\frac{\partial P_i}{\partial \pi_i^1} < 0$, which is a contradiction to $\pi_i^1 \geq \frac{2}{3}$.

Therefore, in equilibrium $\pi_j^1 = \frac{1}{2}$ which implies $\frac{\partial P_i}{\partial \pi_i^2} = -2\pi_j^2 - 2\pi_j^3$, which is zero only if $\pi_j^2 = \pi_j^3 = 0$. The latter implies $\frac{\partial P_i}{\partial \pi_i^1} > 0$, which is a contradiction. Therefore, $\frac{\partial P_i}{\partial \pi_i^2} < 0$ and hence $\pi_i^2 = 0$.

On the other hand, $\pi_j^1 = \frac{1}{2}$ implies $\frac{\partial P_i}{\partial \pi_i^3} = \frac{\partial P_i}{\partial \pi_i^1} = \frac{1}{2} - 3\pi_j^3$. The two cannot be negative, since this would mean an equilibrium with a pure strategy s^0, which is a contradiction. Therefore $\frac{\partial P_i}{\partial \pi_i^3} = \frac{\partial P_i}{\partial \pi_i^1} = 0$, which implies $\pi_j^3 = \frac{1}{6}$. The rest is easy to calculate. Besides the above asymmetric equilibria with pure strategies, the game has a unique equilibrium in mixed strategies, which is symmetric:

$$\text{NE3} \quad \pi_i^* = \left(\tfrac{1}{3}, \tfrac{1}{2}, 0, \tfrac{1}{6}, 0, 0, 0, 0\right) \quad \pi_j^* = \left(\tfrac{1}{3}, \tfrac{1}{2}, 0, \tfrac{1}{6}, 0, 0, 0, 0\right) \tag{A-7}$$

In this symmetric equilibrium, bid $b = 1$ is placed by each player with probability $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, bid $b = 2$ is placed with probability $\frac{1}{6}$, no other bids are placed.

Both players face an expected payoff of zero

End of solution

Solution to Example 3.2

Let three players i, j and y play LUPA with the prize of 4, and bidding costs of 1. Again, consider only bids up to $\mathbf{b} = 3$. Each player may randomize over 8 strategies. Set S^N consists of 512 strategy combinations s. Some asymmetric equilibria can be found by noticing that if one of the players doesn't enter the game, other players play the game from the previous example with respective equilibria, in which one of the players plays strategy s^1 with certainty. The third player has no incentives to enter the game.

$>$From now on we focus on symmetric equilibria in mixed strategies. The marginal expected payoffs of strategies are:

$$\begin{aligned}
\frac{\partial P_i}{\partial \pi_i^0} &= 0\\
\frac{\partial P_i}{\partial \pi_i^1} &= 3\left(\pi^0+\pi^2+\pi^4+\pi^6\right)^2-1\\
\frac{\partial P_i}{\partial \pi_i^2} &= 2\left(\left(\pi^1+\pi^5\right)^2+\left(\pi^0+\pi^4\right)^2\right)-1\\
\frac{\partial P_i}{\partial \pi_i^3} &= \frac{\partial P_i}{\partial \pi_i^1}+2\left(\left(\pi^1+\pi^5\right)^2+2\left(\pi^0+\pi^4\right)\left(\pi^1+\pi^5\right)\right)-1\\
\frac{\partial P_i}{\partial \pi_i^4} &= \left(\pi^0\right)^2+\left(\pi^1\right)^2+\left(\pi^2\right)^2+\left(\pi^3\right)^2-1\\
\frac{\partial P_i}{\partial \pi_i^5} &= \frac{\partial P_i}{\partial \pi_i^1}+\left(\left(\pi^0+\pi^1\right)^2-\left(\pi^0\right)^2+\left(\pi^2+\pi^3\right)^2-\left(\pi^2\right)^2\right)-1\\
\frac{\partial P_i}{\partial \pi_i^6} &= \frac{\partial P_i}{\partial \pi_i^2}+\left(\left(\pi^0+\pi^2\right)^2-\left(\pi^2\right)^2+\left(\pi^1+\pi^3\right)^2-\left(\pi^1\right)^2\right)-1\\
\frac{\partial P_i}{\partial \pi_i^7} &= \frac{\partial P_i}{\partial \pi_i^3}+\left(2\pi^0\pi^3+2\pi^1\pi^2+2\pi^1\pi^3+2\pi^2\pi^3+\left(\pi^3\right)^2\right)-1
\end{aligned} \tag{A-8}$$

Since pure strategies are not possible in the symmetric equilibrium, $\frac{\partial P_i}{\partial \pi_i^t} \leq 0$, $\forall t = 0..7$.

First note that as soon as no strategy is played in equilibrium as a pure strategy, we obtain $\pi^t < 1$, $\forall t = 0..7$ and hence

$$\left(\pi^0\right)^2+\left(\pi^1\right)^2+\left(\pi^2\right)^2+\left(\pi^3\right)^2 < \left(\pi^0+\pi^1+\pi^2+\pi^3\right)^2 \leq 1$$

which implies $\frac{\partial P_i}{\partial \pi_i^4} < 0$. Therefore, $\pi_i^4 = 0$ in equilibrium. The same reasoning applies to

$$\left(\left(\pi^1\right)^2+2\pi^0\pi^1+\left(\pi^3\right)^2+2\pi^2\pi^3\right) < \left(\pi^0+\pi^1+\pi^2+\pi^3\right)^2 \leq 1$$

and

$$\left(\left(\pi^0\right)^2+2\pi^0\pi^2+\left(\pi^3\right)^2+2\pi^1\pi^3\right) < \left(\pi^0+\pi^1+\pi^2+\pi^3\right)^2 \leq 1$$

and

$$\left(2\pi^0\pi^3+2\pi^1\pi^2+2\pi^1\pi^3+2\pi^2\pi^3+\left(\pi^3\right)^2\right) < \left(\pi^0+\pi^1+\pi^2+\pi^3\right)^2 \leq 1$$

which lead to $\frac{\partial P_i}{\partial \pi_i^5} < 0$, $\frac{\partial P_i}{\partial \pi_i^6} < 0$ and $\frac{\partial P_i}{\partial \pi_i^7} < 0$ and therefore to $\pi_i^5 = 0$, $\pi_i^6 = 0$ and $\pi_i^7 = 0$ (since $\frac{\partial P_i}{\partial \pi_i^1} \leq 0$, $\frac{\partial P_i}{\partial \pi_i^2} \leq 0$ and $\frac{\partial P_i}{\partial \pi_i^3} \leq 0$). This reduces the system of inequalities to

$$\begin{aligned}
\frac{\partial P_i}{\partial \pi_i^1} &= 3\left(\pi^0+\pi^2\right)^2-1\\
\frac{\partial P_i}{\partial \pi_i^2} &= 2\left(\left(\pi^1\right)^2+\left(\pi^0\right)^2\right)-1\\
\frac{\partial P_i}{\partial \pi_i^3} &= \frac{\partial P_i}{\partial \pi_i^1}+2\left(\left(\pi^1\right)^2+2\pi^0\pi^1\right)-1
\end{aligned} \tag{A-9}$$

If $\frac{\partial P_i}{\partial \pi_i^1} < 0$ then $\pi^1 = 0$. At the same time $\pi^3 = 0$, and $\pi^0 + \pi^2 = 1$ should hold , which makes $\frac{\partial P_i}{\partial \pi_i^1}$ strictly positive, which is a contradiction.

If $\frac{\partial P_i}{\partial \pi_i^2} < 0$ then $\pi^2 = 0$ and $\left(\pi^1\right)^2 + \left(\pi^0\right)^2 < \frac{1}{2}$. At the same time, $\frac{\partial P_i}{\partial \pi_i^1} = 0$ implies $\pi^0 = \frac{1}{\sqrt{3}}$, hence $\pi^1 < \frac{1}{\sqrt{6}}$ and therefore $\pi^3 = 1 - \pi^0 - \pi^1 > 1 - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} > 0$, which implies $2\left(\left(\pi^1\right)^2 + 2\pi^0\pi^1\right) = 1$. Substituting for $\pi^0 = \frac{1}{\sqrt{3}}$ yields $\left(\pi^1\right)^2 + \frac{2}{\sqrt{3}}\pi^1 - \frac{1}{2} = 0$. This equation has following roots: $\pi^1 = -\frac{1}{\sqrt{3}} \pm \sqrt{\frac{5}{6}} = -\sqrt{\frac{2}{6}} \pm \sqrt{\frac{5}{6}} < 0$, which implies $\pi^1 = \sqrt{\frac{5}{6}} - \sqrt{\frac{2}{6}}$.

Checking for the sum $\left(\pi^1\right)^2 + \left(\pi^0\right)^2$ yields $\left(\sqrt{\frac{5}{6}} - \sqrt{\frac{2}{6}}\right)^2 + \frac{1}{3} = \frac{5}{6} + \frac{2}{6} - 2\sqrt{\frac{10}{36}} + \frac{1}{3} = \frac{9-2\sqrt{10}}{6} < \frac{1}{2}$. Finally, we obtain $\pi^3 = 1 - \sqrt{\frac{5}{6}}$.

If both $\frac{\partial P_i}{\partial \pi_i^2} = 0$ and $\frac{\partial P_i}{\partial \pi_i^1} = 0$ together with $\frac{\partial P_i}{\partial \pi_i^3} < 0$ then we obtain

$$\begin{aligned} \left(\pi^0 + \pi^2\right)^2 &= \frac{1}{3} \\ \left(\pi^1\right)^2 + \left(\pi^0\right)^2 &= \frac{1}{2} \\ \pi^0 + \pi^1 + \pi^2 &= 1 \end{aligned}$$

the first and the third equations yield $\pi^1 = 1 - \frac{1}{\sqrt{3}}$. The second equation yields $\pi^0 = \sqrt{\frac{1+4\sqrt{3}}{18}}$. The first equation yields a contradiction $\pi^2 = \frac{1}{\sqrt{3}} - \sqrt{\frac{1+4\sqrt{3}}{18}} = \sqrt{\frac{6}{18}} - \sqrt{\frac{1+4\sqrt{3}}{18}} < 0$ since $6 < 1 + 4\sqrt{3}$.

Therefore, the game has a unique symmetric equilibrium

$$\text{NE} \quad \pi_i^* = \pi_j^* = \pi_y^* = \left(\left(\frac{1}{\sqrt{3}}\right); \left(\sqrt{\frac{5}{6}} - \sqrt{\frac{2}{6}}\right); 0; \left(1 - \sqrt{\frac{5}{6}}\right); 0; 0; 0; 0\right) \tag{A-10}$$

The expected payoff of players in the symmetric equilibrium is zero

End of solution

Solution to Example 3.3

Similar reasonings as in previous example lead to the following system of inequalities, describing the equilibrium

$$\begin{aligned} \frac{\partial P_i}{\partial \pi_i^1} &= 3\left(\pi^0 + \pi^2\right)^6 - 1 \leq 0 \\ \frac{\partial P_i}{\partial \pi_i^2} &= 2\left(\left(\pi^0 + \pi^1\right)^6 - 6\pi^1\left(\pi^0\right)^5\right) - 1 \leq 0 \\ \frac{\partial P_i}{\partial \pi_i^3} &= \frac{\partial P_i}{\partial \pi_i^1} + 2\left(\left(\pi^0 + \pi^1\right)^6 - \left(\pi^0\right)^6\right) - 1 \leq 0 \end{aligned} \tag{A-11}$$

Assuming $\frac{\partial P_i}{\partial \pi_i^1} < 0$ we obtain $\pi^1 = 0$, which implies $\pi^3 = 0$, hence $\pi^0 + \pi^2 = 1$, which implies $\frac{\partial P_i}{\partial \pi_i^1} > 0$, which is a contradiction. In the following, we assume $\frac{\partial P_i}{\partial \pi_i^1} = 0$.

$\frac{\partial P_i}{\partial \pi_i^2} < 0$ and $\frac{\partial P_i}{\partial \pi_i^3} = 0$ imply $\pi^2 = 0$, hence $\pi^0 = \sqrt[6]{\frac{1}{3}}$, and $\pi^1 = \sqrt[6]{\frac{5}{6}} - \sqrt[6]{\frac{1}{3}}$. Substituting to $\frac{\partial P_i}{\partial \pi_i^2}$ yields $\frac{\partial P_i}{\partial \pi_i^2} > 0$, which is a contradiction.

$\frac{\partial P_i}{\partial \pi_i^2} = 0$ and $\frac{\partial P_i}{\partial \pi_i^3} < 0$ imply $\pi^3 = 0$, hence $\pi^0 + \pi^2 = 1 - \pi^1$, which turns $\frac{\partial P_i}{\partial \pi_i^1} = 0$ into $\pi^1 = 1 - \sqrt[6]{\frac{1}{3}}$. This makes $6\pi^1 \approx 1,004 > 1$. Now consider the difference $\frac{\partial P_i}{\partial \pi_i^2} - \frac{\partial P_i}{\partial \pi_i^3} = \left(\pi^0\right)^6 - 6\pi^1\left(\pi^0\right)^5 = \left(\pi^0\right)^5\left(\pi^0 - 6\pi^1\right)$. Either $\pi^0 = 0$ and $\frac{\partial P_i}{\partial \pi_i^2} = \frac{\partial P_i}{\partial \pi_i^3}$, which is a contradiction, or $\frac{\partial P_i}{\partial \pi_i^2} < \frac{\partial P_i}{\partial \pi_i^3} < 0$ (since $\pi^0 < 1 < 6\pi^1$), which is also a contradiction.

Therefore, in equilibrium $\frac{\partial P_i}{\partial \pi_i^1} = \frac{\partial P_i}{\partial \pi_i^2} = \frac{\partial P_i}{\partial \pi_i^3} = 0$. This can only hold if $\pi^0 = 6\pi^1$ (from the equation $\frac{\partial P_i}{\partial \pi_i^2} = \frac{\partial P_i}{\partial \pi_i^3}$). Condition $\frac{\partial P_i}{\partial \pi_i^2} = 0$ delivers $\pi^1 = \sqrt[6]{\frac{1}{2} \cdot \frac{1}{7^6 - 6^6}}$, and condition $\frac{\partial P_i}{\partial \pi_i^1} = 0$ delivers $\pi^2 = \sqrt[6]{\frac{1}{3}} - 6\sqrt[6]{\frac{1}{2} \cdot \frac{1}{7^6 - 6^6}}$. Probability π^3 is determined by the condition $\sum_{k=0}^{3} \pi^k = 1$. It is easy to prove that all probabilities are strictly positive.

End of solution

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